Denton, L. C. Emmons, C. M. Erikson, J. W. Glover, V. G. Grove, T. H. Hildebrandt, L. A. Hopkins, L. C. Karpinski, W. S. Kimball, Theodore Lindquist, C. E. Love, R. E. Lowney, G. A. Miller, C. N. Mills, H. L. Olson, L. C. Plant, G. Y. Rainich, L. J. Rouse, T. R. Running, R. H. Schoonover, E. W. Schreiber, R. C. Shellenbarger, E. R. Sleight, G. G. Speeker, and C. C. Wagner.

The following papers were presented during the morning meeting:

1. "Report on placement tests at Michigan State College," by Professor S.E. Crowe, Michigan State College.

2. "Cissoids of certain curves," by Mr. W. M. Davis, Albion College (introduced by Professor E. R. Sleight).

3. "On certain inequalities in the theory of definite integrals," by H. J. Bushey, University of Michigan (introduced by Professor J. A. Shohat).

4. "Enrollment in mathematics classes in junior colleges," by Professor R. C. Shellenbarger, Bay City Junior College.

5. "Are our requirements for certification of high school teachers adequate?" by Professor L. C. Emmons, Michigan State College. Discussion by Professor R. W. Clack, Alma College and Professor G. Masselink, Ferris Institute.

6. "A generalization of Cauchy-Riemann equations," by H. C. Chang, University of Michigan (introduced by Professor G. Y. Rainich).

7. "Application of the theory of probability to the evaluation of certain definite integrals," by W. D. Baten, University of Michigan.

At the luncheon at noon in the Ladies' dining room of the Michigan Union, Professor L. C. Plant, Chairman, made a few remarks on the requirements now demanded of teachers of mathematics in the high schools in the State of Michigan. There was a general expression of opinion from those present that a task well worth the efforts of the Association for the next year would be a study of the requirements for teachers of mathematics in the state with recommendations for higher standards.

The following officers were elected for the ensuing year: Professor R. C. Shellenbarger, Chairman; Professor L. A. Hopkins, Secretary-treasurer; Professor C. C. Richtmeyer, Member of the Executive Committee.

The afternoon session of the Association convened at two o'clock when by special invitation Professor G. A. Miller of the University of Illinois delivered an address on the subject, "Profitable use of errors in the history of mathematics." This paper was discussed by Professor L. C. Karpinski of the University of Michigan. An abstract of Professor Miller's follows: He compared the correction of errors in the mathematical literature with the destruction of weeds on a farm and emphasized the fact that such work is constructive as well as destructive. He said that both the student of mathematics and the mathematical investigator have to correct many errors in their views as their insight is enlarged, and hence the training in the correction of errors is valuable to both. Most errors to be corrected arise from misconceptions on the part of the student and not from mistakes in the literature; but a training in the detection of the latter naturally tends towards a clearer view as regards the former. He emphasized the fact

that many of the statements found in historical accounts require modification as our knowledge of the subjects concerned is extended, and he said that just as the fourth figure of a four-place logarithm table is not expected to be always in formal agreement with the fourth figure in a larger table, so a brief historical statement should not always be expected to be in formal agreement throughout with a longer account.

Among the special questions considered by him were the following: Is it true that rational fractions were first used as numbers by the later Greeks and that the quadratic equation was completely solved by them? He pointed out that in the work of Ahmes rational fractions were given as solutions of linear equations and that the number system of the Greeks was not sufficiently extensive to solve the general quadratic equation in the modern sense of this term. He also called in question the term "Hindu-Arabic numerals" and pointed out that some of the latest investigations seem to tend towards a European origin of these numerals. He emphasized the usefulness of the history of mathematics and pointed out that as our knowledge grows we naturally desire more complete historical information, so that the room for work along this line seems to be unlimited. In particular, he noted that we know little in regard to the history of the graphical solution of the quadratic equation when the roots are imaginary.

LOUIS A. HOPKINS, *Secretary*

FOURTEENTH ANNUAL MEETING OF THE OHIO SECTION

The fourteenth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, April 4, 1929, in connection with the meetings of the Ohio College Association. Chairman E. H. Clarke presided over the afternoon session, and Dean T. M. Focke over the evening session.

Fifty-nine persons registered attendance, among whom were the following forty-nine members of the Association: J. P. Albert, R. B. Allen, W. E. Anderson, Grace M. Bareis, I. A. Barnett, H. M. Beatty, H. A. Bender, L. T. Black, H. Blumberg, R. L. Borger, R. S. Burington, W. D. Cairns, E. H. Clarke, R. Crane, P. S. Dwyer, T. M. Focke, C. A. Garabedian, B. C. Glover, H. Hancock, E. J. Hirschler, F. C. Jonah, Margaret E. Jones, E. M. Justin, L. C. Knight, H. W. Kuhn, A. C. Ladner, Anna D. Lewis, C. C. MacDuffee, A. D. Michal, C. C. Morris, J. J. Nassau, Anna B. Peckham, S. E. Rasor, B. H. Redditt, F. W. Reed, C. E. Rhodes, Hortense Rickard, S. A. Rowland, R. A. Sheets, Mary E. Sinclair, G. W. Spenceley, T. S. Peterson, M. O. Tripp, J. H. Weaver, R. B. Wildermuth, F. B. Wiley, C. O. Williamson, J. B. Winslow, C. H. Yeaton,

The following officers were elected for the coming year: Chairman, S. A. Rowland; Secretary-treasurer, Rufus Crane; Member of executive committee. S. E. Rasor; Member of program committee, M. O. Tripp. It is expected that

the next meeting of the Section will be held at the Ohio State University on Thursday, April 3, 1930.

The following nine papers were presented:

1. "Problem solving and mathematical progress," by the Chairman, Professor E. H. Clarke, Hiram College.

2. "Singular solutions of differential equations," by Professor C. C. MacDuffee, Ohio State University.

3. "Least squares as applied to stellar parallaxes" (illustrated), by Professor J. J. Nassau, Case School of Applied Science.

4. "Integral elements of generalized quaternions," by Professor R. A. Sheets, Denison University.

5. "Computation with approximate data," by Professor C. A. Garabedian, University of Cincinnati.

6. "The graphing of irrational functions," by Professor H. A. Bender, University of Akron.

7. "Multiple systems," by Professor Bender.

8. "Elementary and secondary mathematics essential for success in college mathematics," by Mr. A. F. Petersilge, Supervisor of Mathematics in the Shaker Heights School System, Cleveland, Ohio (by invitation).

9. "The teacher's course in mathematics," by Professor M. O. Tripp, Wittenberg College.

Abstracts of these papers follow:

1. Professor Clarke discussed the need for cooperation between mathematicians and engineers and scientists applying mathematics in formulating problem material of value to both, and suggested certain steps looking toward a more useful organization of problem-solving activities. It was also suggested that the Mathematical Association might be rendering a real service to American mathematical progress by fostering such work.

2. If in the usual elementary methods for solving differential equations we watch for irreversible steps, we obtain the singular solutions directly. The theorem that the envelope is the singular solution is of but limited validity, and indeed authorities are far from agreed upon a definition of singular solution. For these reasons a plea is made that in the elementary courses we place less emphasis upon the distinction between particular and singular solutions and more upon the problem of not losing solutions in the process of solving.

3. When the probable error of a series of observations is known the true distribution of the series can be obtained from the formula[1]

$$\epsilon = -h^{-2}[v'(x)/v(x)]$$

where ϵ is the mean square error, h is the precision factor, $v(x)$ and $v'(x)$ the apparent distribution and its derivative, respectively. By means of such formula the true distribution of a series of stellar parallaxes may be obtained. A series of seven hundred stellar parallaxes was taken and was divided into three

[1] Dyson, *A Method of Correcting a Series of Parallax Observations*, M. N., 86, 687.

groups, according to the magnitude of their probable errors. The resulting true distribution of the three groups, however, did not yield the same result and an investigation of the probable errors involved was undertaken. The distribution of the probable errors showed that they follow the Gaussian Law and hence individual probable errors do not measure the precision of the corresponding parallaxes. It was shown that a better measure of the probable errors of the parallaxes may be obtained from the average probable error of the series used.

4. As an example of the integral elements of a rational algebra, the integral generalized quaternions were cited, the method of finding them was outlined and details were given for a special case.

5. Professor Garabedian called attention to the need of a machinery for handling computation with approximate data when logarithms are either not available or not feasible. He proposed a set of practicable and teachable rules which yield results of accuracy compatible with that of the approximate data, and stressed the fact that the devices were such as to ensure also results in agreement with those obtained by logarithms. Professor Garabedian pointed out that it was of especial advantage in the numerical part of trigonometry to have available two equally rigorous and effective methods for computing automatically with approximate data.

6. Too frequently we find in our text-books a discussion of an irrational function after it is rationalized, instead of a discussion of the given function. For example, we find this question: "Find the singular points of $f(x, y)=ay-(x-b)^{5/2}=0$." Since $\partial f/\partial x=-5(x-b)^{3/2}/2$, and $\partial f/\partial y=a$, we see that the curve has a definite tangent at every point on the curve and hence has no singular points. However, in $f(x, y)=a^2y^2-(x-b)^5=0$, $\partial f/\partial x=-5(x-b)^4$, $\partial f/\partial y=2a^2y$, and hence $(b, 0)$ is a singular point of the rationalized function but not of the original function.

7. Triple systems and some of their groups have been known for some time. It is possible to form triple systems on n elements, for $n=6k+1$ or $n=6k+3$, which contain each duad once and only once. Quadruple systems on $n+1$ elements containing each triple once and only once for $n=7$ or 9, and quintuple systems on $n+2$ elements containing each quadruple but once, and sextuple systems on $n+3$ elements containing each quintuple but once, in case $n=9$, have been constructed. In each case the group which leaves the system invariant is given. It is hoped that these higher systems can be established for the more general cases.

8. Success in college mathematics depends upon certain abilities and habits which should be firmly fixed in the child beginning with the first year of arithmetic. Three essentials are: (1) ability to analyze problems, (2) a working knowledge of the technique necessary to solve problems, (3) ability to carry out the process with absolute accuracy and a reasonable degree of speed. These require ability to read, repeated drill on facts and processes, constant practice in mechanical operations. Students should be trained to estimate answers

before beginning work and to look for short cuts, also to check each operation as it is performed. Time should be allowed for checking but not for dawdling. No credit should be allowed unless answer is correct.

9. The college teacher of mathematics should do his part in the professional preparation of those candidates for high school positions who expect to teach mathematics. The teacher's course should be a review of subject matter as well as a course in the pedagogical aspects of the subject. The prerequisites for this pedagogical training should include at least one year of collegiate instruction in mathematics, including analytic geometry.

RUFUS CRANE, *Secretary*

THE EIGHTEENTH MEETING OF THE IOWA SECTION

The eighteenth annual meeting of the Iowa Section of the Mathematical Association of America was held with The Iowa Academy of Science at Parsons College, Fairfield, Iowa, on April 26 and 27, 1929.

The attendance was about forty, including the following eighteen members of the Association: E. W. Chittenden, L. M. Coffin, Julia T. Colpitts, N. B. Conkwright, Marian E. Daniels, Annie W. Fleming, Dora E. Kearney, F. M. McGaw, J. F. Reilly, H. L. Rietz, B. D. Roberts, Fred Reusser, E. R. Smith, J. S. Turner, L. E. Ward, C. W. Wester, Roscoe Woods, C. C. Wylie. Sessions were held Friday afternoon and Saturday morning, the Section chairman, Professor Wester, presiding. Dinner was enjoyed together Friday evening, at which time Professor Colpitts gave a very interesting account of "The Bologna Congress." Officers were elected for 1929–1930 as follows: Chairman, E. W. Chittenden, University of Iowa; Vice-chairman, L. M. Coffin, Coe College; Secretary-treasurer, J. F. Reilly, University of Iowa. The program consisted of seventeen papers, as follows:

1. "On Stieltjes integrals and their geometric interpretation," by E. W. Chittenden, University of Iowa.

2. "Fitting an n-dimensional surface to an equation of the form: $X_1 = F(X_2, X_3, X_4, \cdots, X_n)$," by A. E. Brandt, Iowa State College, by invitation.

3. "A note on Leibnitz's theorem for the differentiation of the product of two functions," by J. F. Reilly, University of Iowa.

4. "A torsion problem in curvilinear coordinates," by E. W. Anderson, Iowa State College, by invitation.

5. "Approximate solution of the problem of Mr. Anderson," by D. L. Holl, Iowa State College, by invitation.

6. "Remark on the existence theorem for the differential equation $dy/dx = f(x, y)$," by N B. Conkwright, University of Iowa.

7. "An examination of the remainder after n terms of Fourier series from the point of view of contour integration," by N. B. Conkwright.

8. "Each diagonal of a parallelopiped passes through the centroids of two diagonal triangles," by J. S. Turner, Iowa State College.

9. "A property of coaxial circles," by J. S. Turner.

10. "The Bologna Congress," by Miss Julia Colpitts, Iowa State College.

11. "Mathematics and common sense," by C. W. Wester, Iowa State Teachers College.

12. "Some properties of a class of curves," by Roscoe Woods, University of Iowa.

13. "Some mistakes of the amateur mathematicians," by E. R. Smith, Iowa State College.

14. "A comparison of the distribution curves of variance and of standard deviation," by H. L. Rietz, University of Iowa.

15. "An evaluation of periodicals from a mathematician's point of view," by E. S. Allen, Iowa State College.

16. "The diurnal path of the end of a shadow," by C. C. Wylie, University of Iowa.

17. "The interest rate on a loan repaid by installments," by C. C. Wylie.

Abstracts of these papers follow:

1. Professor Chittenden's paper contained an elementary discussion of the properties of Stieltjes integrals in terms of a three dimensional representation.

2. In his paper, Professor Brandt set forth a method of fitting a surface to an n-dimensional equation, using least squares and assuming all relations linear. The true nature of the functional relation is approximated by graphic methods and the coefficients determined by least squares.

3. In his paper, Professor Reilly developed a formula for expressing the n-th integral of the product of two functions in terms of the integrals of the functions, analogous to Leibnitz's formula for differentiation.

4. Professor Anderson showed how the torsion problem for a right prism whose boundary consists of arcs of orthogonal parabolas could be effected by means of conjugate cylindrical coordinates, characterized by the relation $2Z = W^2$. Methods were indicated for computing the torque and the stresses.

5. Using Raleigh's approximate method, which consists in assuming a suitable stress function, containing an arbitrary parameter and satisfying all the static boundary conditions on the contour, Professor Holl exhibited a solution of the above torsion problem. He determined the arbitrary parameter by the condition that the energy of deformation is equal to the work of the torquing couple.

6. In his first paper, Professor Conkwright made an elementary application of the method of successive approximations to establish the uniqueness of the solution of the differential equation on a certain interval.

7. In his second paper, Professor Conkwright indicated a method for expressing the remainder in certain cases as the remainder after n terms of the Fourier series for $f(x) \equiv x$, plus a contour integral of relatively simple form.

8. In his first paper, Professor Turner proved that in a tetrahedron $OABC$, if planes through A, B, C, parallel to the opposite faces meet in O', then OO' will pass through the centroid of the triangle ABC.

9. Professor Turner, in his second paper, showed that if E is a limiting point of two circles, and if any chord AB of one circle cuts the other in C, then EC meets the circle AEB in a point which lies on a fixed circle.

10. Professor Colpitts, in her paper, gave first a few points on the history of mathematics at Bologna. A book on that subject, which was given to each member at the Congress, was shown. In it is a photograph of a page from a manuscript in the University of Bologna in which Scipione dal Ferro, a professor at Bologna, gives correctly the rule for the solution of the cubic. This was some years before Cardan published the "Ars Magna."

At the opening meeting of the Congress, Professor Birkhoff as representative of the foreign mathematicians gave the reply to the address of welcome. General meetings were held in mornings and sectional meetings in afternoons. The papers were in either Italian, French, German, or English.

About eight hundred active and three hundred associate members were at the Congress, making the attendance considerably larger than at any other.

The social life was very enjoyable. Members were entertained at three receptions, a very fine concert, and a luncheon. One day was given up to excursions. The one to Ravenna was particularly enjoyable.

At the last meeting which was held at Florence, Professor Birkhoff spoke on some mathematical elements of art. The next Congress will be in Switzerland in 1932.

12. A class of curves was suggested to Professor Woods by the parametric equations of a curve generated when one curve is rolled upon another. They may be defined as the common intersection of a system of concentric circles and a system of lines passing through their common center. The author represents these curves parametrically and derives the conditions for some of their singularities.

13. Professor Smith's departmental mail bag gives a very good glimpse into the activities of a large number of amateur mathematicians. This paper, based on a number of unsolicited communications, shows the attitude of many persons towards a group of famous problems in mathematics and the errors which are generally made by such persons in attempting to obtain solutions.

14. In this paper by Professor Rietz, a discussion is given by both graphical and numerical methods of the distribution curves for variances and standard deviations obtained from small samples. It is obvious that the two sets of curves are very closely related because the standard deviation is the square root of the variance, but the differences in general appearance of the curves is such that for small samples the practical statistician would almost surely feel that the distribution of standard deviations tends more towards symmetry and normality than the distribution of variances. Moreover, this feeling is justified by a comparison of the measures of skewness of the curves.

15. This paper by Professor Allen was a report on the investigation of the relative importance of magazines for mathematicians, based upon a complete study of the citations made in the 1928 volumes of nine of the leading mathematical journals of the world.

16. In his first paper, Professor Wylie stated that neglecting parallax, refraction, and the sun's motion in declination, the paths, shown by a model of an equatorial telescope, are all conic sections. If the sun goes below the horizon during the 24 hours, the path is a hyperbola including the straight line at the equinoxes as a limiting case. This means that in the torrid and temperate zones the path is always hyperbolic.

In polar regions, the path is a hyperbola if the sun sets, a parabola if the midnight sun is just on the horizon, and an ellipse if the sun is above the horizon throughout the 24 hours. At the pole itself the paths are circular, the circle becoming infinite in radius at the equinoxes.

17. Assuming that each interest payment should be figured on the money the borrower has in use, and that each installment should include payment of the interest to date, and a balance to be applied to reduction of the principal, it follows that the interest rate on an installment loan is that given by annuity tables.

Professor Wylie discussed a few plausible and rather common methods of figuring interest rates on such loans, giving rates from about half the correct up to double the correct. A new approximate interest formula was presented, by which the actual interest rate can be figured almost as quickly as the simple interest when no tables are at hand. The formula is given in a paper which is now ready for publication.

J. F. REILLY, *Secretary*

THE TENTH ANNUAL MEETING OF THE ILLINOIS SECTION

The tenth annual meeting of the Illinois Section of the Mathematical Association of America was held May 3 and 4, 1929, at Carthage College, Carthage, Illinois under the chairmanship of Professor A. E. Gault of Bradley Polytechnic Institute, Peoria, Illinois.

The attendance was thirty-one including the following twenty-four members of the Association: Edith I. Atkin, S. F. Bibb, O. E. Brown, C. E. Comstock, A. E. Gault, R. M. Ginnings, Lois Griffiths, Mabel Heren, Byron Ingold, Nellie M. Johnston, B. W. Jones, E. C. Kiefer, W. C. Krathwohl, Martha P. McGavock, C. I. Palmer, H. P. Rogers, Theresa Renner, C. H. Smiley, G. T. Sellew, H. E. Slaught, C. J. Stowell, Mildred Taylor, C. A. Van Velzer, Alice Winbigler.

The time outside of the meetings was fully employed by the entertainment which was provided through the courtesy and cooperation of Professor Van Velzer and Carthage College. An automobile trip to the thirty million dollar dam and power plant at Keokuk, Iowa, was of unusual interest. At the dinner Friday evening Dr. Wickey, president of Carthage College, was an honored guest and gave a message of welcome. During the meeting the women members of the Association were guests of the College at Denhart Hall, the women's

dormitory, and the college invited the entire group to a luncheon after the Saturday morning meeting.

In the absence of the secretary-treasurer, Professor C. N. Mills, who is on leave of absence at the University of Michigan, Miss Edith I. Atkin served as secretary-treasurer pro tem. The following officers were elected for 1929–1930: Chairman, C. A. Van Velzer, Carthage College; Vice-chairman, H. B. Curtis, Lake Forest College; Secretary-treasurer, C. N. Mills, Illinois State Normal University. The meeting of 1930 will be held at Lake Forest College, May 2 and 3.

The following program was presented:

1. "Four years of unified mathematics," by Professor C. J. Stowell, McKendree College.

2. "The determination of cometary orbits," by Dr. C. H. Smiley, University of Illinois.

3. "High school mathematics as an index to college ability," by Miss Theresa M. Renner, Registrar, Blackburn College.

4. "The representation of integers by forms in number theory," by Dr. B. W. Jones, University of Chicago.

5. "The mathematics major for the bachelor's degree," by Professor Byron Ingold, Culver-Stockton College, Canton, Missouri.

6. "A theorem on polygonal numbers," by Dr. Lois W. Griffiths, Northwestern University

7. "Organization of freshmen mathematics at Armour Institute of Technology," by Dr. C. W. Krathwohl, Armour Institute.

8. "Sets of numbers with equal sums of like powers," by Mr. O. E. Brown, Graduate Student, University of Chicago.

9. "Euclid and the boy," by Professor Martha P. McGavock, Rockford College.

Abstracts of these papers follow:

1. Professor Stowell presented a history of the reorganization of mathematics courses at McKendree College, whereby in four years the traditional freshman and sophomore courses were replaced by the unified plan, with the Griffin series for texts. The old and the new plans were compared, with the following principal conclusions:

(a) The unified course, as exemplified by the Griffin texts, succeeds best with students well grounded in entrance requirements; for other students more algebra review is desirable.

(b) The unified course presents certain definite advantages along the line of concreteness and correlation, as well as some routine advantages.

(c) The unified course is to be preferred to the traditional courses for students whose principal interest will be in the applications of mathematics.

(d) The traditional courses are to be preferred for students majoring in mathematics for the sake of teaching and research in this subject.

(e) On account of a growing demand from those intending to be teachers

the traditional separate courses are to be resumed for a time at McKendree College.

2. Dr. Smiley pointed out that the field of orbit methods is one which has interested many of the greatest mathematicians. Two of the three methods extensively used today are modifications of those introduced by Gauss and Laplace. There are both mathematical and physical difficulties inherent in the problem of orbit determination. This is especially true in the case of cometary orbits. However, because of the importance of comets in the question of cosmogony, it is very desirable that the improvement and development of orbit methods continue.

3. Miss Renner's paper was a discussion of the general qualities required for doing satisfactory work during the first year of college and a study of the high school courses, by correlating students' grades in different courses with ratings at the end of the first semester in college, to determine what courses are indicative of success in college. Is the correlation between high school mathematics and college ratings higher than that of other courses or of the general average of the entire high school course? Study to cover a period of five years and has not been completed.

4. Dr. Jones gave a brief survey of some of the more important results on the representation of integers by forms. Classical theorems concerning the integers represented by the sum of two, three, and four squares were given as well as the number of ways in which a given integer may be represented in one of these forms. He gave some of the outstanding results of Professor L. E. Dickson and his students in investigating the integers represented by ternary quadratic forms and forms of higher degree. Mention was made of some methods of finding the number of representations by certain ternary forms.

5. In this paper Professor Ingold stressed the need of uniformity among schools in regard to general, departmental, and group requirements for all majors, and the impossibility of constructing a uniform "mathematics major" under the present conditions. He also pointed out that there should be some flexibility in the major itself to allow selection of courses suited to the subsequent work of the individual student. Prospective teachers in college and high school, engineers, and those who aspire to university professorships and research work were all considered in the proposed "mathematics major."

6. If m is a positive integer, the polygonal numbers of order $m+2$ are 0 and certain positive integers, the values of $p(x)=x+\frac{1}{2}m(x^2-x)$ for $x=0, 1, 2, \cdots$. The Fermat theorem for polygonal numbers of order $m+2$ states that if A is a positive integer then there are $m+2$ polygonal numbers of order $m+2$, namely $p_1, \cdots, p_{m+2}$, such that $A=p_1+\cdots+p_{m+2}$; by the theorem of this paper it is not necessary to use $m+2$ different polygonal numbers of order $m+2$. For example, if $m=3$ the polygonal numbers of order 5 are the pentagonal numbers $0, 1, 5, 12, 22, 35, \cdots$. The Fermat theorem states that if A is a positive integer there are five pentagonal numbers $p_1, \cdots, p_5$ such that $A=p_1+\cdots+p_5$; by the theorem of this paper, if A is a positive integer there are four pentagonal

numbers $p_1, \cdots, p_4$ such that $A = p_1 + p_2 + p_3 + 2p_4$. Generally, let $n, a_1, \cdots, a_n$ be positive integers, and $p_1, \cdots, p_n$ be polygonal numbers of order $m+2$. The function $f = a_1p_1 + \cdots + a_np_n$ is said to represent A when $p_1, \cdots, p_n$ can be chosen so that $A = a_1p_1 + \cdots + a_np_n$. In this paper $m \geqq 3$, $a_1 + \cdots + a_n \leqq m+2$, and there are found all values of $a_1, \cdots, a_n$ such that f represents every positive integer A. This theorem is more powerful than the Fermat theorem, since it implies the latter and also reduces the problem to one in fewer than $m+2$ variables.

7. This paper describes a plan, used by the Armour Institute of Technology for a number of years, which automatically separates the freshmen who need a further review in mathematics from those better fitted to carry on college work, and which tends to remedy an inadequate preparation.

8. Some 63 published papers on this subject are treated by Professor Dickson in his history. This paper is an expository treatment of those results. The notation $x_i \overset{m}{=} y_i (i = 1, \cdots, n)$ is used to represent the conditions

$$\sum_{i=1}^{n} x_i^j = \sum_{i=1}^{n} y_i^j \quad (j = 1, \cdots, m).$$

We may roughly divide the paper into three parts: I. Development of the theory, a collection of theorems, most of which show how to set up solutions or how to pass from one to another; II. Exhibitions of the more elegant formulas; III. Exhibition of formulas giving all solutions of $x_1, x_2, x_3 \overset{2}{=} y_1, y_2, y_3$ in integers and all rational solutions of $x_1, x_2, x_3, x_4 \overset{3}{=} y_1, y_2, y_3, y_4$.

9. Professor McGavock suggests that the answer given by Euclid to the boy asking the use of mathematics was not a wise one, and she raises the question whether the reply that Hipparchus would have given would not be a better guide to the modern teacher.

EDITH I. ATKIN, *Secretary pro tem.*

REMARKS ON LINEAR EQUATIONS[1]

By AUBREY J. KEMPNER, University of Colorado

The theory of a system of m linear equations with n unknowns ($m \leqq n$) has been completely developed. The following points, which have come up in courses on higher Algebra, refer only to details.

I. Consider n equations with n unknowns, non-homogeneous. According to the theory, if the rank r of the matrix = rank R of the augmented matrix, the system has solutions. If $r \neq R$, then necessarily $r = R - 1$, and the system has no solutions.

For example, for $n = 3$, only the following familiar cases can arise:
$r = 3$, $R = 3$: three planes intersecting in one point; one solution.

[1] Presented at the Meeting of the Rocky Mountain Section, Greeley, Colorado, April 12–13, 1929.

$r=2$, $R=3$: the point of intersection has receded to infinity, the planes form the faces of a three sided prism; no finite solution.

$r=2$, $R=2$: the cross section of the prism reduces to a point, the three planes pass through a line; every point on this line represents a solution, we say that there are ∞^1 solutions.

$r=1$, $R=2$: the common line of intersection recedes to infinity, the planes are parallel but not coinciding; no solution.

$r=1$, $R=1$: the parallel planes all coincide, every point in the plane represents a solution; we say that we have ∞^2 solutions.

$r=0$, $R=1$: the triple plane recedes to infinity; no solution.

$r=0$, $R=0$: trivial, satisfied by all points of space; we have ∞^3 solutions.

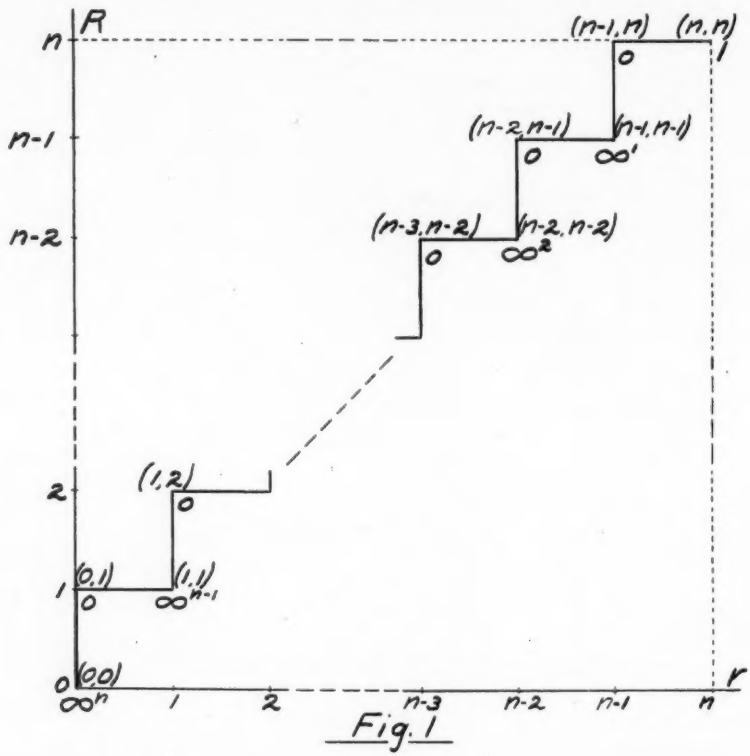

Fig. 1

We see that with increasing degree of degeneracy of the problem we have 1, 0, ∞^1, 0, ∞^2, 0, ∞^3 solutions. This is characteristic of the behavior in the general case of a non-homogeneous system of n equations with n unknowns. This well known state of affairs may be graphically represented by Fig. 1

II. A peculiarly satisfying aspect of the general theory is the uniformity of

procedure in all cases ($m \gtreqless n$, homogeneous or non-homogeneous). However, at the very end, this uniformity seems to be given up and the discussion is allowed to follow slightly differing lines in the homogeneous and the non-homogeneous case. The following extremely simple modification of the usual treatment may perhaps not be considered too artificial.

The customary discussion is followed up to the point of reduction of the problem to a non-homogeneous system (A) or a homogeneous system (B), each of n equations with n unknowns. In case A assume the system consistent, i.e., there are solutions. In both cases assume $r(=R)<n$. A: Let first $r=n-1$. There exists more than one solution, say $(x_1', x_2', \cdots, x_n') = (x_i')$, as we write for brevity,[2] and (x_i''). Then for any given k_1, k_2,

$$(1) \qquad \left(\frac{k_1x_i' + k_2x_i''}{k_1 + k_2}\right) = \left(\frac{k_1x_1' + k_2x_1''}{k_1 + k_2}, \cdots, \frac{k_1x_n' + k_2x_n''}{k_1 + k_2}\right), \quad k_1 + k_2 \neq 0,$$

is also a solution. If $r=n-1$, there are no further solutions. The solutions are the coordinates of all points of a line in n-dimensional space through which all the ($n-1$ dimensional) planes pass. However, since the line of intersection extends to infinity, there will also be infinite solutions. These correspond to $k_1+k_2=0$. We will carry them along, although it is understood that the finite solutions correspond to $k_1+k_2\neq 0$.

Let next $r=n-2$. Now, besides (1), there are still other solutions. Let (x_i''') be such an one. Then

$$\left(\frac{k_1x_i' + k_2x_i'' + k_3x_i'''}{k_1 + k_2 + k_3}\right),$$

k_1, k_2, k_3 arbitrary, but $k_1+k_2+k_3\neq 0$ represents the totality of finite solutions, while $k_1+k_2+k_3=0$ corresponds to infinite solutions.

This continues in an obvious fashion:

For $r=n-\mu$,

$$(2) \qquad \left(\frac{k_1x_i' + k_2x_i'' + \cdots + k_{\mu+1}x_i^{(\mu+1)}}{k_1 + k_2 + \cdots + k_{\mu+1}}\right),$$

$k_1, \cdots, k_{\mu+1}$ arbitrary, represents all finite solutions for $\sum k\neq 0$, and the infinite solutions for $\sum k=0$.

B. Let first $r=n-1$: There exists a solution (x_i') besides the trivial solution $(0, 0, \cdots, 0)$, which does not permit a geometrical interpretation in homogeneous coordinates. Then, (k_1x_i') is also a solution for all k_1. There are no other solutions. For $k_1=0$ we have the trivial solution which is usually rejected.

Next let $r=n-2$: Now there is at least one solution besides (k_1x_i'). Let such a solution be (x_i''). Then $(k_1x_i'+k_2x_i'')$ is also a solution, and all solutions are contained in this form. In homogeneous coordinates, $(k_1x_i'+k_2x_i'')$ represents the same point as does (1), provided $k_1+k_2\neq 0$. For $k_1+k_2=0$, we still

[2] Similarly, throughout.

assume that in homogeneous coordinates $(k_1x_i' + k_2x_i'')$ and (1) represent the same point. For $k_1 = k_2 = 0$, we have the trivial solution.

This continues in an obvious fashion:

For $r = n - \mu$, the general solution is given by $(k_1x_i' + \cdots + k_\mu x_i^{(\mu)})$, which is, in homogeneous coordinates, the same point[3] as

$$\left(\frac{k_1x_i' + \cdots + k_\mu x_i^{(\mu)}}{k_1 + \cdots + k_\mu}\right),$$

provided $\sum k \neq 0$. For $\sum k = 0$, we still assume it to represent the same point. If each k is 0, we assume it to represent the trivial solution.

The general solutions for A and B, respectively, $r = n - \mu$ in both cases, are now given by

A:
$$\left(\frac{k_1x_i' + \cdots + k_{\mu+1}x_i^{(\mu+1)}}{k_1 + \cdots + k_{\mu+1}}\right),$$

B:
$$\left(\frac{k_1x_i' + \cdots + k_\mu \cdot x_i^{(\mu)}}{k_1 + \cdots + k_\mu}\right).$$

The only discrepancy left is the lag in the number of terms in each coordinate in (B) as compared with (A). This can be remedied by saying:

Theorem: *The general solution for the non-homogeneous case, $r = n - \mu$, is given by*

$$\left(\frac{k_1x_i' + \cdots + k_{\mu+1}x_i^{(\mu+1)}}{k_1 + \cdots + k_{\mu+1}}\right).$$

Here, $k_1 + \cdots + k_{\mu+1} \neq 0$ corresponds to the totality of finite solutions, while $\sum k = 0$ takes care of the infinite solutions.

The general solution for the homogeneous case, $r = n - \mu$, is given by the same expression, with $k_{\mu+1} = 0$, $x_i^{(\mu+1)}$ arbitrary. When $k_1 + \cdots + k_{\mu+1} = 0$, we ignore the denominator. When all $k = 0$, we have the trivial solution.

The assumptions made concerning the interpretation of the fractions with vanishing denominators appear natural under the assumptions of the problem.

III. One may enquire what corresponds in a system of linear equations to the theorem on an equation in one unknown, $a_0x^n + \cdots + a_n = 0$: each root is a function of $a_0, \cdots, a_n$, continuous in $a_1, \cdots, a_n$, and continuous for all $a_0 \neq 0$.

Consider a system $a_{i1}x_1 + \cdots + a_{in}x_n = b_i$, $i = 1, 2, \cdots, n$, not all $b_i = 0$, and of non-vanishing determinant Δ. One finds immediately:

Theorem: *Each x_i is a linear function of $b_1, \cdots, b_n$.*

Each x_i is a fractional linear function of each a_{ij}.

A simple geometrical interpretation of these results is the following:

Let b_1 be the only one of the quantities a_{ij}, b_i which is permitted to vary.

[3] It is quite customary to consider two solutions of homogeneous equations, $(x_1, \cdots, x_n)$ and $(cx_1, \cdots, cx_n)$, equivalent, on account of the identical geometrical representation of the two when interpreted as homogeneous coordinates in $n-1$ dimensional space.

Plot, in the same system of cordinates, all of the x_i against a b_1-axis. Fig. 2 then explains itself:

Letting next b_1 and b_2 vary independently, and using a three-dimensional system of coordinates, plotting each x_i as a function of b_1, b_2, one finds of course that each x_i moves in a plane. Et cetera.

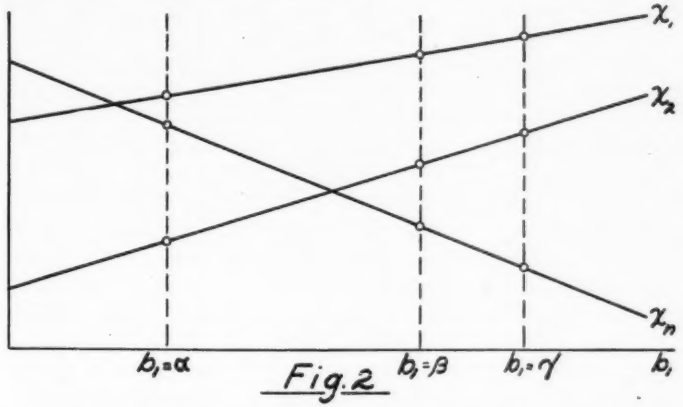

Fig. 2

If we let a_{11} be the only one of the quantities a_{ij}, b_i which is allowed to vary, and plot all x_i in the same system of coordinates as functions of a_{11}, we obtain a system of hyperbolae with horizontal asymptotes and a common vertical asymptote. A very simple transformation (of type $u=(Ca_{11}+D)^{-1}$, C, D constant) transforms all of the hyperbolae into straight lines.

The first theorem may be employed in the solution of a non-homogeneous system (n equations n unknowns, $\Delta \neq 0$). There must be at least one non-vanishing first minor; assume $\Delta_{1n} \neq 0$. Select arbitrarily two values x_n', x_n'' for x_n, and solve the two systems of $n-1$ equations in $n-1$ unknowns:

$$\sum a_{mi}x_i = b_m - a_{mn}x_n', \qquad m = 2, \cdots, n,$$
$$\sum a_{mi}x_i = b_m - a_{mn}x_n'', \qquad m = 2, \cdots, n.$$

Since these two systems differ only in the constant terms, the solution of either of them by determinants (expanding in first minors according to the column containing the terms on the right side) gives the solution of the other system with very little additional work. Let $(x_1', \cdots, x_n')$, $(x_1'', \cdots, x_n'')$, respectively, be solutions of these two systems; let $b_1' = a_{11}x_1' + \cdots + a_{1n}x_n'$, $b_1'' = a_{11}x_1'' + \cdots + a_{1n}x_n''$, and let k_1, k_2 be two numbers such that $b_1 = (k_1b_1' + k_2b_2')/(k_1+k_2)$. Then

$$\left(\frac{k_1x_1' + k_2x_1''}{k_1 + k_2}, \quad \frac{k_1x_n' + k_2x_n''}{k_1 + k_2}\right)$$

is the solution of the given system.

The solution of the original problem is thus reduced to the solution of two (very closely related) systems of $n-1$ equations in $n-1$ unknowns. This method may be extended by letting more than one of the b_i vary simultaneously (instead of only b_1, as above), but the result seems of doubtful practical value.

A FORERUNNER OF MASCHERONI

By FLORIAN CAJORI, University of California

The Italian Lorenzo Mascheroni who published in 1797 a well known work on the *Geometry of the Compasses*,[1] in which all constructions are effected without a ruler and by the use only of compassses, was anticipated by 125 years, as is now first shown, by a Danish writer, Georg Mohr[2] whose book, *Euclides Danicus* of 1672, the Royal Danish Scientific Society at Kopenhagen has just published in *facsimile* and also in translation into German. The book was overlooked by mathematicians, nowithstanding the fact that there appeared two editions in 1672, one in Danish, the other in Dutch. There is nothing to indicate that Mascheroni had any knowledge of Mohr's book. The two worked independently. The *Euclides Danicus* in Dutch covers 36 pages and is a much smaller book than that of Mascheroni. It consists of two parts, the first part containing 54 constructions in Euclid's *Elements*, effected by the use of only the compasses. The last few problems call for the construction of a figure similar to a given figure and equal in area to another. An easy problem is the following: Given the line BA, to find the end point of a line twice as long. Draw an arc with A as center and AB as radius. Starting at B, apply to this arc, using the compasses, BA three times successively as a chord; the final intersection on the arc is the required point E of the straight line BAE. The second part of Mohr's book gives 24 constructions of various selected problems, ending with a rather involved problem on the erection of a sun dial.

Mohr's book is mentioned by some bibliographers but without a hint as to the nature of its content. From its title one might surmise that it was an edition of Euclid's *Elements*. Leibniz refers to him in a letter to Oldenburg (May 12, 1676) as "Georgius Mohr Danus, in geometria et analysi versatissimus." More is known of him than is indicated by the editors of the 1928 edition of his book. Cantor[3] refers to Mohr's trip to England and thence to France where, about 1676, he met Leibniz and informed him that Collins was in possession of infinite series for arc sin x and sin x. Before this, Oldenburg had mentioned Mohr, in a letter to Leibniz of September 30, 1675, as one well versed in algebra,

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[1] Lorenzo Mascheroni, *Geometria del compasso* (Pavia 1797, Palermo, 1901.)

[2] Georg Mohr, *Euclides Danicus*, Amsterdam, 1672. Mit einem Vorwort von Johannes Hjelmslev und einer deutschen Uebersetzung von Julius Pál. Udgivet af det Danske Videnskabernes Selskab, København, Hovedkommissionaer: Andr. Fr. Høst & Søn, Kgl. Hof-Boghandel 1928.

[3] Moritz Cantor, *Vorlesungen über Geschichte der Mathematik*, vol. III, 2nd ed., p. 179.

and mechanics, who left with John Collins a manuscript on roots of $A+\sqrt{B}$ written in Dutch. This tract was forwarded to Leibniz. Born in 1640 in Copenhagen, Mohr went to Holland as a young man and did not return to Denmark except for a short time in 1682.[4]

It may be proper to introduce here some historical remarks. Ever since the dawn of abstract geometry, mathematicians have taken delight in limiting the kind and number of instruments to be used in effecting geometric constructions. Apparently urged by the ideal of simplicity and economy in instrumental equipment, the Greeks ordained that in the science of geometry only an ungraduated straight edge and compasses shall be used. This limitation has given rise to some of the most interesting and famous discussions in geometry and analysis—on the squaring of the circle, trisection of an angle, duplication of a cube, and the inscription of regular polygons in a given circle. From time to time, further instrumental restrictions have been made in the interest of speculative geometry and mathematical recreation. Inspired perhaps by a remark of Pappus, the Arabic scholar Abu'l Wefa[5] of the tenth century, in constructing the corners of the regular polyhedrons on a circumscribed sphere, set himself the condition that all constructions be effected with a ruler and a *single* opening of the compasses. The German painter Albrecht Dürer and the Italian mathematicians of the sixteenth century, including Benedetti and Tartaglia,[6] effected many constructions under these limitations. J. V. Poncelet[7] in 1822 and Jakob Steiner[8] in 1833 went a step further and showed that all constructions possible with a straight edge and compasses can be effected also by the use of a straight edge, and a circle fixed in position and drawn once for all. In 1890 A. Adler[9], of Vienna, went still further and demonstrated that all these constructions can be made by the use of only an ordinary ruler with two parallel straight edges, or only a ruler in the form of a right angle, or only a ruler in the form of a fixed acute angle. If we take cognizance also of the fact that all constructions possible by the straight edge and compasses can be effected by the compasses alone, as was shown by Mohr, Mascheroni, and later writers, then the remarkable result stares us in the face that all Euclidean constructions can be made with *any one* of the four ordinary instruments of geometric construction taken singly; viz., the compasses, or the ruler with parallel straight edges, or the ruler with a right angle, or the ruler with an acute angle.

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[4] See G. Eneström, Bibliotheca Mathemateca, (3) vol. 10 (1909–1910), pp. 71, 72; vol. 12 (1911–1912), p. 77.

[5] See Woepcke, Journal Asiatique, (5) vol. 5 (1855), pp. 241, 352–358.

[6] See Moritz Cantor, *op. cit.*, vol. II, 2nd ed., 1913, p. 566.

[7] J. V. Poncelet, *Traité des propriétés projectives des figures*, Paris, 1822, p. 187–190.

[8] Jakob Steiner, *Ueber die geometrischen Constructionen ausgeführt mittels der geraden Linie und eines festen Kreises*, Berlin, 1833.

[9] A. Adler, *Ueber die zur Ausführung geometrischer Constructionsaufgaben zweiten Grades notwendigen Hülfsmittel*, Wiener Sitzungsberichte d. Akademie d. Wiss., Math.-Naturw. Classe, vol. 99 (1890), Abth. IIa, pp. 846–859.

TRANSFORMATIONS ON CUBIC EQUATIONS

By RAYMOND GARVER, University of California at Los Angeles

The general Tschirnhaus transformation on the roots of a cubic equation (which we may take in the reduced form $x^3+a_2x+a_3=0$) may be written

$$y=m\,(x^2+k_2\,x+k_3)\;; \tag{1}$$

the general linear fractional or homographic transformation is

$$y=\frac{ax+b}{cx+d}\qquad (ad-bc\neq 0). \tag{2}$$

It is well-known that any transformation of form (2) is reducible to one of form (1); in fact, any rational algebraic transformation on the roots of an equation of the nth degree is reducible to an integral or Tschirnhaus transformation. Conversely, it is sometimes important to determine whether or not a Tschirnhaus transformation is equivalent to a rational transformation of some particular form, for example, whether or not transformation (1) on the roots of a cubic is equivalent to transformation (2).

It seems to be generally stated that this last equivalence does exist. Burnside and Panton say, in their *Theory of Equations* (4th edition, vol. 2, page 167): "Whence it appears that the most general [Tschirnhaus] transformation of a root of a cubic may be reduced to a homographic transformation." In Niewenglowski's *Cours d'Algèbre* (5th edition, vol. 2, page 314) is found the statement: "Il résulte de là que la transformation rationnelle la plus générale, relativement à l'équation du 3e degré, est la transformation homographique." And other similar references might be given.

These statements are not entirely correct, however, since there are certain exceptional Tschirnhaus transformations on cubic equations which are not equivalent to homographic transformations. The purpose of this paper is to study these transformations, which do not seem to be mentioned in the literature.

That such exceptional transformations do exist, it is not difficult to show. Assume that transformation (1) is given, and that we wish to determine a, b, c, d (if possible) so that (2) will be equivalent to it. Equate the right-hand members of (1) and (2), clear of fractions, and reduce by using $x^3+a_2x+a_3=0$. This gives:

$$ax+b=m(ck_2+d)x^2+m(ck_3+dk_2-ca_2)x+m(dk_3-ca_3). \tag{3}$$

If (3) is to be an identity in x, corresponding coefficients on the two sides must be equal and there are three equations to satisfy. The first gives the condition $d=-ck_2$, and the others then reduce to

$$a=mc(k_3-k_2^2-a_2),\qquad b=mc(-k_2k_3-a_3). \tag{4}$$

In general, c may be chosen at will ($\neq 0$), and a, b, d are then determined.

Furthermore, the steps in the process can be reversed to show that the two transformations are actually equivalent.

However, *in case k_2 is a root of the cubic* $x^3+a_2x+a_3=0$, a valid linear fractional transformation is not obtained. For the denominator $c(x-k_2)$ will vanish when x takes on the value k_2. Also, as the reader can easily verify with the aid of (4), $ad-bc=0$ for this case. Hence we may state:

THEOREM 1. *The Tschirnhaus transformation (1) on the roots of the cubic* $x^3+a_2x+a_3=0$ *is not reducible to a linear fractional transformation in case* k_2 *is a root of the cubic.*

The question next arises as to what is the effect of applying one of these exceptional transformations. This is answered in

THEOREM 2. *A cubic equation can be transformed by a Tschirnhaus transformation into an equation with a double root.*[1]

For if we apply[2] $y=x^2+x_3x+k_3$ we shall have

$$\begin{aligned} y_1 &= x_1^2 + x_3x_1 + k_3 = -x_1x_2 + k_3, \\ y_2 &= x_2^2 + x_3x_2 + k_3 = -x_1x_2 + k_3, \\ y_3 &= x_3^2 + x_3x_3 + k_3 = 2x_3^2 + k_3. \end{aligned} \tag{5}$$

A similar result obviously follows if k_2 is taken equal to x_1 or x_2.

Let us now compute the transformed equation in y explicitly. I have considered this question in a previous paper[3] and shall not reproduce the details of the work here. Briefly, if m in (1) is taken equal to 1, and if k_3 is chosen as $2a_2/3$, the transformed equation will be $y^3+A_2y+A_3=0$, where

$$\begin{aligned} -2A_2 &= \sum y^2 = \frac{2}{3}a_2^2 - 2k_2^2a_2 - 6k_2a_3, \\ -3A_3 &= \sum y^3 = 3a_3^2 + \tfrac{2}{9}a_2^3 + 3k_2a_2a_3 + 2k_2^2a_2^2 - 3k_2^3a_3. \end{aligned} \tag{6}$$

For the present case, where k_2 satisfies the equation $k_2^3+a_2k_2+a_3=0$, the second equation of (6) becomes

$$-3A_3 = 6a_3^2 + \tfrac{2}{9}a_2^3 + 6k_2a_2a_3 + 2k_2^2a_2^2. \tag{7}$$

But we also know that the transformed equation has a double root, say r. And from its form the third root is $-2r$. Hence the equation is

$$y^3 - 3r^2y + 2r^3 = 0. \tag{8}$$

We may obtain an expression for r which does not involve k_2 by eliminating the latter from the equations $A_2=-3r^2$, $A_3=2r^3$, where A_2 and A_3 are replaced by

[1] This result is clearly not obtainable with a linear fractional transformation.

[2] We hereafter take $m=1$. The roots of the given cubic are x_1, x_2, x_3; those of the transformed cubic, y_1, y_2, y_3.

[3] This Monthly, vol. 34 (1927), pp. 521–525.

their values as given in (6). The result of the elimination is a cubic equation in r, which can be written as

$$r^3 - a_2 r^2 - (D/27) = 0 \quad (D = -4a_2^3 - 27a_3^2), \tag{9}$$

or as

$$(r - \tfrac{2}{3}a_2)^2(r + \tfrac{1}{3}a_2) = -a_3^2 . \tag{10}$$

We should expect r to satisfy a cubic, its three values corresponding to the three possible values of k_2.

Equations (9) and (10) are of interest because they may be used in the derivation of a number of well-known properties of the cubic equation. First, let r represent the double root $y_1 = y_2$ of (5); we then have, replacing y_3 by $-2r$, which is correct if $k_3 = \frac{2}{3}a_2$, and dividing the third equation of (5) by -2,

$$r = -x_3^2 - \tfrac{1}{3}a_2 . \tag{11}$$

The other two values of r are similar, with x_3 replaced by x_1 and x_2 in turn. Hence we may say that the transformation $r = -x^2 - \frac{1}{3}a_2$ leads from the given equation $x^3 + a_2x + a_3 = 0$ to the transformed equation (10). If we put $s = -(r + \frac{1}{3}a_2) = x^2$, (10) reduces at once to $s(s + a_2)^2 = a_3^2$; and we have found by an indirect method the equation whose roots are the squares of those of $x^3 + a_2x - a_3 = 0$.

Next, consider the case in which the original cubic in x already has a double root. Then two of the three possible values of k_2 will lead to a transformed equation (8) with a triple root. But (8) clearly has a triple root only if $r = 0$. Referring to (9), we obtain the well-known condition $D = 0$ on the coefficients of the given cubic. The argument can be modified easily to show that this condition is also sufficient to insure that the given cubic have a double root.

Another known result of a similar nature follows immediately from a consideration of (5) and (11). Assume in (5) that $x_1 = x_3$. Then $y_1 = y_2 = y_3$, and, as above, r must be zero. Then from (11), $x_1^2 = x_3^2 = -\frac{1}{3}a_2$; and we have proved the theorem that if the equation $x^3 + a_2x + a_3 = 0$ has a double root it is $\pm\sqrt{(-a_2/3)}$.

Finally, with the aid of elementary calculus, we can supplement the result of the paragraph before the last and obtain the usual relations between the value of D and the nature of the roots of the given cubic. If higher powers of x are eliminated between $x^3 + a_2x + a_3 = 0$ and $r = -x^2 - \frac{1}{3}a_2$, the transformation which, as pointed out above, leads to (9) or (10), we have $x = a_3/(r - \frac{2}{3}a_2)$, which can be interpreted as a rational transformation leading from (9) or (10) back to the original equation in x. But the main point is that x is determined *rationally* in terms[4] of r; hence we can investigate the nature of the roots of $x^3 + a_2x + a_3 = 0$ by examination of (9).

To this end, we consider the cubic polynomial[5]

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[4] At least in case $r \neq \frac{2}{3}a_2$. And this condition is satisfied if $a_3 \neq 0$.

[5] The y in (12) has no connection with the y used before in this paper. We are simply considering the graph of (12) in an (r, y) axis-system. The student is supposed to be familiar with the essential features of the graph of a cubic polynomial.

(12) $$y = r^3 - a_2r^2 - (D/27).$$

Three cases must be distinguished: (1) a_2+, $D-$; (2) a_2-, $D-$; (3) a_2-, $D+$. In all cases (12) has its maximum and minimum values at $P_1(0, -D/27)$ and $P_2(\frac{2}{3}a_2, a_3^2)$. In case (1) both of these points are above the r-axis, and the graph of (12) will cut the r-axis in one and only one point, which will of course be to the left of the y-axis. The same thing is true in case (2); the only reason for distinguishing it from (1) is that now P_2 is to the left of P_1, and is the maximum point, whereas in (1) it is to the right of P_1 and is the minimum point. Combining these two results, we may say that when D is negative equation (9) and hence the equation $x^3+a_2x+a_3=0$ has but one real root. In case (3) the minimum point P_1 is below the r-axis, while the maximum point P_2 is still above it; a construction of the graph shows that it now cuts the r-axis in 3 real points. That is, when D is positive, equation (9) and hence the equation $x^3+a_2x+a_3=0$ has three real roots. The other possibility, $D=0$, has already been treated above by a purely algebraic method.

ON NOVEL MAGIC SQUARES

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1. Magic Squares are defined as squares that are divided into n^2 smaller squares in which are inserted numbers in arithmetical progression (usually the numbers from 1 to n^2) so that the sum of the n numbers in any row, column, or diagonal is constant. These squares have been known from the earliest times and in many countries magic properties have been ascribed to them. In recent years there have been many investigations concerning these squares and the methods of their construction, and some interesting and ingenious squares have been constructed by different writers. However, in all these squares with the various combinations and permutations of the numbers within them, their common characteristic remains unaltered: the sum of the n numbers in the rows and columns and the diagonals is always the same constant for the given square.

In my research into these squares I came to notice in a certain combination of the numbers a peculiar feature that led me afterwards to construct a new type of magic squares with different properties than those already known. I present here the new kind of squares and the method of their construction, as I think that they may be of interest and value as a contribution towards the research into the properties of magic squares.

2. *Features of the New Square.* In Fig. 1 is represented a magic square of 7 of the ordinary type where the numbers from 1 to 49 are so arranged that the sum of the seven numbers in any row, column, or diagonal is equal to 175. These numbers can be rearranged in many different ways so that the constant sum will always remain the same, according to the usual definition of magic squares.

In Fig. 2 is represented also a square of 7, but of the new type. There the numbers from 1 to 49 are inserted, and have the following property: the sum of *any* four numbers that lie in a sub-square of four is constant, and in this instance is equal to 93. For example: $4+15+32+42=93$, $42+30+8+13=93$, $42+32+11+8=93$, etc. So also is the sum of the four numbers at the four corners of every even sub-square or rectangle; $46+28+18+1$, $28+43+8+14$, $17+47+22+7$, etc., all have the same constant sum, 93.

35	23	18	13	1	45	40
4	48	36	31	26	21	9
22	17	12	7	44	39	34
47	42	30	25	20	8	3
16	11	6	43	38	33	28
41	29	24	19	14	2	46
10	5	49	37	32	27	15

Fig. 1

26	46	28	44	24	43	27
17	4	15	6	19	7	16
40	32	42	30	38	29	41
10	11	8	13	12	14	9
47	25	49	23	45	22	48
3	18	1	20	5	21	2
33	39	35	37	31	36	34

Fig. 2

32	28	46	14	18	7	39
17	22	3	36	31	43	10
34	26	48	12	20	5	41
16	23	2	37	30	44	9
35	25	49	11	21	4	42
15	24	1	38	29	45	8
33	27	47	13	19	6	40

Fig. 3

This *four-number-constant* is the basic characteristic of this type of squares. In all squares of this kind, of whatever size they may be, only the sum of these four numbers is looked for. But unlike the ordinary magic square, this sum, while it remains constant for a given arrangement of the numbers in a given square, is not constant for the given square. In the same square the numbers can be arranged in a different order so as to give a different constant sum. This is shown in Fig. 3, where the four-number-constant is 99, instead of 93 as in Fig. 2. It will be proved later that in every odd square it is possible to obtain *nine* different constants. These are the two basic features of the new type of squares that distinguishes them from the old ones.

3. *Method of Construction.* In the arithmetical series $1, 2, 3, \cdots, 2m-2, 2m-1$, the sum of any two terms equidistant from the two ends respectively is equal to $1+(2m-1)$, or $2m$, and the sum of any two such pairs is evidently $4m$. When we divide the same series into two groups of m and $m-1$ terms, viz. $1, 2 \cdots m-1, m$; and $m+1, m+2, \cdots, 2m-2, 2m-1$; then the sum of each pair in the first group will be $m+1$, and the sum of any such pair in the second group will be $3m$. The sum of one pair of the first group and one pair of the second group will then be $4m+1$. Again, if we divide the original series into two groups of $m-1$ and m terms respectively, viz. $1, 2, \cdots, m-2, m-1$; and $m, m+1, \cdots, 2m-2, 2m-1$; then the sum of any pair in the first group will be m and that of the second group $3m-1$; and the sum of two such pairs will be $4m-1$. Thus we arrive at three different sums for two pairs of numbers in the same series, viz. $4m+1$, $4m$, and $4m-1$. Since for our purpose it is imposed that the terms in each sub-group shall constitute an arithmetical progression in themselves, it is obvious that no other subdivision of the original series into two such groups is possible.

We shall illustrate the foregoing by an example: In the upper row of Fig. 4 the numbers 1 to 7 are written in any order. In the second row the numbers are written to supplement the upper numbers so as to form pairs as described above, i.e., their sum shall be 8. These two rows are the *basic* rows of the square. In the rows that follow below, the *basic* rows are repeated cyclically. It can be seen at a glance that any two adjacent numbers (or numbers that are separated by an *even* number of rows) in any column form a pair and hence the sum of any four numbers that lie at the four corners of a square will be constant, because they are composed of two pairs with a constant sum.

3	7	6	4	5	2	1
5	1	2	4	3	6	7
3	7	6	4	5	2	1
5	1	2	4	3	6	7
3	7	6	4	5	2	1
5	1	2	4	3	6	7
3	7	6	4	5	2	1

Fig. 4

2	7	1	5	3	6	4
3	5	4	7	2	6	1
2	7	1	5	3	6	4
3	5	4	7	2	6	1
2	7	1	5	3	6	4
3	5	4	7	2	6	1
2	7	1	5	3	6	4

Fig. 5

5	3	4	2	6	1	7
6	1	7	2	5	3	4
5	3	4	2	6	1	7
6	1	7	2	5	3	4
5	3	4	2	6	1	7
6	1	7	2	5	3	4
5	3	4	2	6	1	7

Fig. 6

To construct the square shown in Fig. 5, we divide the numbers 1 to 7 into two groups of 4 and 3 numbers respectively, i.e. into the two series 1, 2, 3, 4 and 5, 6, 7; and we write the numbers of the first group in any order in the *odd* places in the first row, and the numbers of the second group in the *even* places in the same row, also in any order. In the second row the numbers of each group are written to supplement the corresponding numbers of the same group. The rows are then repeated cyclically as in the preceding example. Here, too, every four-number-square is composed of two pairs with a constant sum, but different from that in Fig. 4. The square in Fig. 6 is constructed in the same manner as that of Fig. 5, but with the sub-groups 1, 2, 3 and 4, 5, 6, 7 instead, which results in a different four-number-constant.

These are the three fundamental constructions of this type of squares. Evidently the numbers may be written in columns instead of in rows; and we shall denote them as vertical and horizontal basic squares respectively.

In order to form a resultant square similar to that in Fig. 2, two of the basic squares are combined together, one horizontal and one vertical, each having one of the three basic constructions. This gives rise to *nine* different constant sums, as was pointed out before-hand.

In Fig. 7 is represented such a composite square. To obtain Fig. 8 from Fig. 7, the left hand digit in each cell of Fig. 7 is diminished by unity and the resulting number in the cell is thought of as a number in the septenary scale of notation. Fig. 8 then is derived by translating from the septenary to the decimal

scale. For example: the upper left hand corner cell of Fig. 7 contains the digits 25; diminishing the left hand digit by unity gives 15; 15 thought of as a number in the septenary scale is equal to $1\times7+5=12$ in the decimal scale; so 12 is the entry in the upper left hand cell of Fig. 8. Similarly 33 of Fig. 7 is translated into 17 of Fig. 8; 24 is translated into 11; etc. The same composite square (Fig. 7) can be used to construct another resultant square by changing the right hand digit instead of the left hand one in each cell of Fig. 7 before translating to the decimal scale. This second square is shown in Fig. 9.

25	33	24	32	26	31	27
76	51	77	52	75	53	74
15	43	14	42	16	41	17
56	71	57	72	55	73	54
35	23	34	22	36	21	37
66	61	67	62	65	63	64
45	13	44	12	46	11	47

Fig. 7

12	17	11	16	13	15	14
48	29	49	30	47	31	46
5	24	4	23	6	22	7
34	43	35	44	33	45	32
19	10	18	9	20	8	21
41	36	42	37	40	38	39
26	3	25	2	27	1	28

Fig. 8

30	17	23	10	37	3	44
42	5	49	12	35	19	28
29	18	22	11	36	4	43
40	7	47	14	33	21	26
31	16	24	9	38	2	45
41	6	48	13	34	20	27
32	15	25	8	39	1	46

Fig. 9

From our foregoing investigations about the sums of the different pairs in the arithmetical series $1, 2, 3 \cdots 2m-2, 2m-1$, it can be easily shown that in any square of n numbers in a side, where n is equal to $2m-1$, the nine four-number-constants will be $2(n^2+1)$, $2(n^2+1)\pm1$, $2(n^2+1)\pm n$, and $2(n^2+1)\pm n\pm1$. It is also evident from the method of construction that each square is subject to a great number of permutations without altering its constant. Thus the mode of construction of the square in Fig. 4 gives rise to $(2m-1)!$ permutations. And if the composite square is a combination of two such squares, the resulting square may then be formed in $[(2m-1)!]^2$ different ways. The number of permutations in squares like those in Figs. 5 and 6 is much less, as there are only $m!(m-1)!$ permutations. In practice, however, some of these combinations will prove impossible.

Squares of this type may also be constructed so as to have added features besides those already described. We shall give only two examples: In Fig. 10, the sum of any two pairs that lie in a broken horizontal line equals 52, the constant of the square. For instance: $18+9+24+1=52$, $18+9+14+11=52$, $9+17+1+25=52$, etc. The same will hold also when the "break" in the line is filled with an even number of rows; thus: $13+12+7+20=52$, $17+10+3+22=52$, etc.

13	12	14	11	15
18	9	17	10	16
23	2	24	1	25
8	19	7	20	6
3	22	4	21	5

Fig. 10

9	4	3
2	5	8
7	6	1

Fig. 11

The square of 3, shown in Fig. 11, has the peculiar feature that, besides every square of four numbers having the constant sum of 20, every line or diagonal can be made to have this same sum by taking double its middle number. Thus: $9+1+(2\times5)=20$, $9+7+(2\times2)=20$, etc.

4. *The Even Square.* Those who are acquainted with magic squares know that even squares present certain difficulties and that more caution is necessary in their construction. While it is comparatively easy to construct odd squares, it is not so with even squares. The same difficulty I found in attempting to construct even squares of the new type. The difficulty consists in this, that in what ever way we divide into two sub-groups, of m terms each, the series $1, 2, 3 \cdots 2m-1, 2m$, the sum of two pairs from these groups will always be $4m+2$. This renders the impossibility of construction of even squares with different constants, which is the most interesting feature of this new kind of squares. Every even square has thus only one four-number-constant, $2(n^2+1)$. It may be added in passing, that this mode of construction in some instances makes the square "magic" in the usual sense also, the rows and columns, excepting the diagonals, having a constant sum.

In Fig. 12 the four-number-constant is 130, and the sum of the numbers in each row or column is equal to 260.

1	63	3	61	8	58	6	60
56	10	54	12	49	15	51	13
17	47	19	45	24	42	22	44
40	26	38	28	33	31	35	29
57	7	59	5	64	2	62	4
16	50	14	52	9	55	11	53
41	23	43	21	48	18	46	20
32	34	30	36	25	39	27	37

Fig. 12

The lack of variety mentioned above is overcome when we consider the square as composed of several sections. This at once opens the road to many possibilities of combinations. The square is then divided into sections, each having its own constant. These constants may be different from each other, or

all alike. The square of 6 in Fig. 13 is divided into 4 sub-squares with different constants viz. 68, 70, 80, and 82; whereas all four sub-squares in Fig. 14 have their constants alike, viz. 75.

33	5	31	6	34	2
13	17	15	14	16	18
3	35	1	36	4	32
7	29	9	26	10	30
21	23	19	24	22	20
25	11	27	8	28	12

Fig. 13

2	29	3	25	4	30
21	23	20	24	22	19
26	5	27	1	28	6
9	35	8	36	10	31
14	17	15	13	16	18
33	11	32	12	34	7

Fig. 14

It is the selection of the proper numbers for the sub-division that will prove one's ability in the construction of these squares.

THE FUNDAMENTAL THEOREM IN RIGID KINEMATICS

By LOUIS BRAND, University of Cincinnati

In this note a simple proof is given for the existence of the angular velocity vector ω in the most general motion of a rigid body by the use of the Plücker coordinates of a line. If O and P are two fixed points of a rigid body in motion, we then prove the fundamental theorem,

$$\mathbf{v}_P = \mathbf{v}_O + \omega \times \mathbf{r} \quad (\mathbf{r} = \overrightarrow{OP}), \tag{1}$$

giving the velocity distribution in the body.

1. *Plücker Coordinates.* Let α be a line in space and P any one of its points of position vector $\mathbf{r}$ referred to a fixed origin O_1. Then if $\mathbf{a}$ is a vector along α and $\mathbf{a}' = \mathbf{r} \times \mathbf{a}$, the vectors $\mathbf{a}, \mathbf{a}'$ (taken in this order) are called the Plücker coordinates of the line. Obviously $\mathbf{a}'$ is the same for all choices of P. The line itself is denoted by $(\mathbf{a},\mathbf{a}')$; its equation is $\mathbf{r} \times \mathbf{a} = \mathbf{a}'$.

In order that $(\mathbf{a},\mathbf{a}')$ represent a line it is necessary that

$$\mathbf{a} \cdot \mathbf{a}' = \mathbf{a} \cdot \mathbf{r} \times \mathbf{a} = 0. \tag{2}$$

Conversely if $\mathbf{a} \cdot \mathbf{a}' = 0$, the line is determined by choosing

$$\mathbf{r} = \frac{\mathbf{a} \times \mathbf{a}'}{\mathbf{a}^2}; \quad \text{then} \quad \mathbf{a}' = \mathbf{r} \times \mathbf{a}.$$

A necessary and sufficient condition that the lines $(\mathbf{a},\mathbf{a}')$ and $(\mathbf{b},\mathbf{b}')$ be coplanar is that

$$\mathbf{a} \cdot \mathbf{b}' + \mathbf{b} \cdot \mathbf{a}' = 0. \tag{3}$$

If $\mathbf{a}' = \mathbf{r} \times \mathbf{a}$, $\mathbf{b}' = \mathbf{s} \times \mathbf{b}$, this follows at once from the identity

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{r} - \mathbf{s}) = \mathbf{a} \cdot \mathbf{b}' + \mathbf{b} \cdot \mathbf{a}'.$$

2. Let O be a point fixed in a rigid body moving in any manner, and $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ non-coplanar position vectors to three of its points. Since the body is rigid $\mathbf{r}_i^2$ and $\mathbf{r}_i \cdot \mathbf{r}_j$ do not vary with the time. Hence on differentiating with respect to the time,

$$\mathbf{r}_i \cdot \dot{\mathbf{r}}_i = 0 \qquad (i = 1,2,3),$$
$$\mathbf{r}_i \cdot \dot{\mathbf{r}}_j + \mathbf{r}_j \cdot \dot{\mathbf{r}}_i = 0 \qquad (i,j = 1,2,3).$$

From (2) and (3) above we see that $(\mathbf{r}_1, \dot{\mathbf{r}}_1)$, $(\mathbf{r}_2, \dot{\mathbf{r}}_2)$, $(\mathbf{r}_3, \dot{\mathbf{r}}_3)$ are the Plücker coordinates of three lines which are coplanar in pairs. Since $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ were chosen as three non-coplanar vectors, these lines cannot be coplanar; at every instant they must therefore be concurrent in some point J. Then if $\boldsymbol{\omega} = \overrightarrow{OJ}$, the second Plücker coordinates of the lines are given by

$$\dot{\mathbf{r}}_i = \boldsymbol{\omega} \times \mathbf{r}_i \qquad (i = 1,2,3). \tag{4}$$

Now the position vector $\overrightarrow{OP}$ to any point of the body may be written

$$\mathbf{r} = l\mathbf{r}_1 + m\mathbf{r}_2 + n\mathbf{r}_3$$

where l, m, n, are constant scalars. Hence

$$\dot{\mathbf{r}} = l\dot{\mathbf{r}}_1 + m\dot{\mathbf{r}}_2 + n\dot{\mathbf{r}}_3,$$

or in view of (4)

$$\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}. \tag{5}$$

The vector $\boldsymbol{\omega}$ defined as above is called the *angular velocity* of the body. At any instant it is independent of the choice of O. For if we write

$$\mathbf{s} = \overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ} = \mathbf{r}_P - \mathbf{r}_Q,$$
$$\dot{\mathbf{s}} = \dot{\mathbf{r}}_P - \dot{\mathbf{r}}_Q = \boldsymbol{\omega} \times \mathbf{r}_P - \boldsymbol{\omega} \times \mathbf{r}_Q = \boldsymbol{\omega} \times \mathbf{s}.$$

3. We are now in position to prove the fundamental theroem (1). If O_1, O are points fixed in space and in the body respectively,

$$\overrightarrow{O_1P} = \overrightarrow{O_1O} + \overrightarrow{OP}.$$

The time derivatives of these three vectors are precisely the corresponding terms of (1).

4. If $\boldsymbol{\omega}$ were the time derivative of a vector function, this function might be appropriately defined as the "angle." However, in general, no such vector function exists, so that the term "angular velocity" for $\boldsymbol{\omega}$ is to some extent a misnomer.

If $\mathbf{i}, \mathbf{j}, \mathbf{k}$ and $\mathbf{i}_1, \mathbf{j}_1, \mathbf{k}_1$ are two sets of orthogonal unit vectors fixed in the

body and in space respectively, we may regard the rotation dyadic (tensor of the second order)

$$R = ii_1 + jj_1 + kk_1$$

as a species of generalized angle which defines the orientation of the body at any instant. In view of (5) its "velocity" is

$$\frac{dR}{dt} = \omega \times R.$$

ON COMPLEX VALUES OF A REAL PARAMETER

By BOYD C. PATTERSON, Hamilton College

1. *Introduction.* In a recent paper in the American Journal of Mathematics,[1] the writer made use of the fact that if given a curve parametrically expressed by a complex function of a real parameter, $x = x(\mu)$, we may allow μ to take complex as well as real values. For such values of μ the parametric equation gives not points on the curve but image points with respect to the curve. The equation of the curve is obtained by eliminating the real parameter from $x = x(\mu)$ and its conjugate equation $\bar{x} = \bar{x}(\mu)$. The result will be a self-conjugate equation[2] in x and $\bar{x}$:

$$f(x, \bar{x}) = 0.$$

It is the purpose of this paper to illustrate by a more elementary application than that in the above mentioned paper the effectiveness of such a parametric representation in certain instances and at the same time to further elucidate the general principle under consideration.

2. *Parametric equation of a circle.* It is known that the transformation $x = e^w$ maps the complex w-plane upon the complex x-plane in such a way as to send straight lines parallel to the real axis of the w-plane into half-rays proceeding from the origin of the x-plane; and straight lines parallel to the axis of imaginaries of the former into circles with centers at the origin of the latter.[3]

Putting $w = u + iv$ and giving u the constant value u_0 we have a parametric representation of a circle with radius $\rho_1 = \exp u_0$ and center at the origin:

$$x = \rho_1 e^{iv}.$$

Writing $x - c_1$ for x in this equation we obtain

$$x = \rho_1 e^{iv} + c_1 \tag{2,1}$$

a circle of radius ρ_1 and center c_1.

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[1] Vol. 50 (1928), pp. 553–568.

[2] The condition that the equation be self-conjugate is equivalent to the condition that the coefficients be real in a cartesian equation. Professor Morley has made extensive use of these coordinates with much success. See also Winger's *Projective Geometry*, D. C. Heath & Co., 1923, p. 324.

[3] For a complete discussion see E. J. Townsend's *Functions of a Complex Variable*, p. 126.

Elimination of the parameter v from (2,1) and its conjugate

$$\bar{x} = \rho_1 e^{-iv} + \bar{c}_1 \tag{2,2}$$

gives us the equation of the circle:

$$x\bar{x} - \bar{c}_1 x - c_1\bar{x} + c_1\bar{c}_1 - \rho_1^2 = 0. \tag{2,3}$$

Upon replacing $\bar{x}$ by $\bar{y}$ in (2, 3), we get

$$x\bar{y} - \bar{c}_1 x - c_1\bar{y} + c_1\bar{c}_1 - \rho_1^2 = 0; \tag{2,3'}$$

and the equation is no longer self-conjugate and does not represent a curve. It is rather a relation between two points, x and y, of the x-plane. For a given x, y is determined by (2,3′) and conversely. Each of these points is termed the image of the other with respect to the circle (2,3) and for that reason we call (2,3′) the image equation of the circle. Such an equation is of much broader significance than the usual cartesian equation for while the latter simply tells what points lie on the curve the former, that is the image equation, is a relation on all points of the plane. In the case at hand it is evident if we write (2,3′) in the form,

$$(x - c_1)(\bar{y} - \bar{c}_1) = \rho_1^2,$$

that x and y are inverse points with respect to a circle of radius ρ_1 and center c_1—that is, the circle (2,1) or (2,3).

In the foregoing, v has been restricted to real values. If we extend its range to complex values, writing for that purpose $v=\alpha+i\beta$, equation (2,1) takes the form

$$x = \rho_1 e^{i\alpha}\lambda^{-1} + c_1, \tag{2,1'}$$

where $\lambda = e^{\beta}$ is real, while (2,2) becomes

$$\bar{y} = \rho_1\lambda e^{-i\alpha} + \bar{c}_1. \tag{2,2'}$$

Here we have put $\bar{y}$ instead of $\bar{x}$ as in (2,2) for equation (2,2′) is not the conjugate of (2,1′). It is easily verified that the x and y of these two equations are inverse points of the circle (2,1).

3. *Application.* Given two circles, C_1 and C_2, to find their common inverse points. Let the circles have respectively the radii ρ_1 and ρ_2 and the centers c_1 and c_2. The image equation of the circle C_2 is

$$x\bar{y} - \bar{c}_2 x - c_2\bar{y} + c_2\bar{c}_2 - \rho_2^2 = 0. \tag{3,1}$$

Any pair of points inverse with respect to the circle C_1 is given by the two equations

$$x = \rho_1 e^{iv} + c_1, \quad \bar{y} = \rho_1 e^{-iv} + \bar{c}_1, \tag{3,2}$$

where the parameter v is complex. (Put $v=\alpha+i\beta$ and equations (2,1′) and (2,2′)

are recovered.) Substituting the values of x and y as given by (3,2) in equation (3,1) we have a function of $\exp iv$, namely,

$$\rho_1(\bar{c}_1 - \bar{c}_2)e^{iv} + [(c_1 - c_2)(\bar{c}_1 - \bar{c}_2) + \rho_1^2 - \rho_2^2] + \rho_1(c_1 - c_2)e^{-iv}.$$

The zeros of this function are the values of v which give with the aid of (3,2) the common inverse points. The distance between the centers of C_1 and C_2 is $(c_2 - c_1)$, a complex number which we shall write $\delta \exp i\phi_0$. With this simplification the above reduces to the solution of

$$[\exp i(v - \phi_0)]^2 - 2A[\exp i(v - \phi_0)] + 1 = 0,$$

where

$$A = (\delta^2 + \rho_1^2 - \rho_2^2)/2\delta\rho_1.$$

Therefore

$$\exp i(v - \phi_0) = A \pm (A^2 - 1)^{1/2}$$

and

$$v = \phi_0 - i \log [A \pm (A^2 - 1)^{1/2}].$$

Since ϕ_0 is the direction angle of the line of centers and is constant, the above values of v obviously depend on A. We consider the two cases:

(i) When $A^2 > 1$. The quantity $A + (A^2 - 1)^{1/2}$ is a real number, say λ_0. Then $A - (A^2 - 1)^{1/2} = 1/\lambda_0$ and $v = \phi_0 \pm i \log \lambda_0$. Hence $\alpha = \phi_0$; $\beta = \pm \log \lambda_0$, giving the two common inverse points[4]

$$(3,3) \qquad x_0 = \rho_1\lambda_0^{-1} \exp i\phi_0 + c_1; \quad y_0 = \rho_1\lambda_0 \exp i\phi_0 + c_1.$$

(ii) When $A^2 \leqq 1$. If $A^2 < 1$ we may put

$$A \pm (A^2 - 1)^{1/2} = \cos\theta_0 \pm i \sin\theta_0 = \exp(\pm i\theta_0),$$

for the left member is a complex number of absolute value 1. Here $\theta_0 = \cos^{-1}A$ is the angle included between the line of centers and that radius of C_1 drawn from its center to the point of intersection of the two circles. For v in this case has the values $v = \phi_0 \pm \theta_0$, which are real and are hence parameters of points on the circle. If $A^2 = 1$ we find $\theta_0 = 0$ or π and the circles are tangent.

The results of (i) are of course the most important. A little calculation will show that $(x_0 - c_1)$ and $(y_0 - c_1)$ have the same direction angle as the line of centers, namely ϕ_0. Hence x_0 and y_0 lie on the line of centers. Furthermore the circle C_0 on x_0y_0 as diameter has a radius equal to $\rho_1(A^2 - 1)^{1/2}$ and center $c_0 = \rho_1 e^{i\phi_0}A + c_1$. The absolute value of the segment $(c_0 - c_1)$ is $\rho_1 A$. With this information we apply the trigonometrical law of cosines and find C_0 to be orthogonal to C_1 and to C_2.

4. *Conclusion.* The construction of these points from the viewpoint of

[4] (3.3) gives the values of x_0, y_0 for $\beta = +\log \lambda_0$. If the negative sign is taken, x_0 and y_0 are interchanged.

synthetic geometry probably has been recalled already by the foregoing analysis. The easily proved theorem that "every line through the center of a given circle cuts all orthogonal circles in two points inverse with respect to the given circle" and its converse that "all circles through inverse points with respect to a circle cut that circle orthogonally" enable us to restate the problem as follows: Given two circles, to construct a third orthogonal to both. The intersections of this orthogonal circle with the line of centers of the given circles are the points sought.

For a further application of this principle the reader is refered to the paper first cited. The analysis therein is particularly effective and as is frequently the case gives results not possible by synthetic geometry.

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

DISCUSSIONS

I. INTERPRETATION OF SOLUTIONS FOR THE TIME OBTAINED BY INTERPOLATION IN THE MATHEMATICS OF INVESTMENT

By W. L. HART, UNIVERSITY OF MINNESOTA

1. *Introduction.* In an elementary application of the mathematics of investment leading to an exponential equation, the solution can usually be obtained either by logarithmic methods or by interpolation in standard tables. It frequently happens that the interpolation method leads to results which have definite and useful interpretations, while the so-called exact logarithmic methods give only approximate results, as judged by current business practice. The purpose of this note is to emphasize the useful interpretations which can be given to the solutions obtained by interpolation in certain types of problems. The most important result considered is in connection with the amortization of debts; a convenient method[1] is formulated for determining the final partial payment in case a debt is discharged by equal periodic payments of specified size.

2. *Fundamental equations for linear interpolation.* Suppose that a table is available giving the values of a real-valued function $f(n)$ for integral values of n. Consider solving for the unknown n in an equation

$$f(n) = c, \tag{1}$$

where c is a given constant. If $c = f(k)$, where k is some integer, the solution of

[1] This method is implicity described in *Politische Arithmetik*, by Emil Foerster (Walter de Gruyter and Company, Berlin), a book of small circulation in this country. The method deserves wider appreciation than it has been given heretofore.

equation 1 can be found by mere inspection of the given table of values of $f(n)$. If c lies between two table entries, $f(k)$ and $f(k+1)$, and if we solve for n by linear interpolation between these values, we conclude that $n=k+h$, where

$$h = \frac{[c - f(k)]}{[f(k+1) - f(k)]}. \tag{2}$$

On placing $c=f(n)=f(k+h)$ in equation (2), we see that our solution by interpolation assumes that, if n has the form $n=k+h$ where k is an integer and $0<h<1$, then

$$f(k+h) = f(k) + h[f(k+1) - f(k)]. \tag{3}$$

That is, by interpolation we obtain the exact solution of equation (1) under the assumption that equation (3) defines $f(n)$ when n is not an integer.

Since the right member of equation (3) is linear in h, the geometric assumption equivalent to equation (3) is that, for values $n=k+h$ between $n=k$ and $n=k+1$, the graph of $f(n)$ is a straight line joining the points $[k, f(k)]$ and $[(k+1), f(k+1)]$. Under condition (3), the whole graph of $f(n)$ would be a broken line with its corners at the points where n has integral values.

3. *An application to compound interest.* Consider the equation

$$A = P(1+i)^n, \tag{4}$$

where A is the compound amount at the end of n conversion periods, if a principal P is invested at the rate i per conversion period. Suppose that A, P, and i are given and that we solve for the unknown time n by linear interpolation in a table of values of $(1+i)^k$ for integral values of k. By use of equation (3) with $f(n)=P(1+i)^n$, we find that our interpolated value of n is the exact solution of equation (4) under the assumption that the amount A at the end of $(k+h)$ periods is given by

$$A = P(1+i)^k + hiP(1+i)^k.$$

$$A = P(1+i)^k(1+hi). \tag{5}$$

We can translate equation (5) into the following directions for finding the amount A at the end of a given time, $(k+h)$ periods, which is not a whole number of periods: *First, find the compound amount at the end of the last whole period contained in the given time; then, accumulate the resulting amount for the remaining time at simple interest at the given nominal rate.*

The value of A given by equation (5) is sometimes called the practical compound amount, and is usually employed as the working approximation to the compound amount in current business whenever the time of investment is not an integral number of conversion periods. Hence, we have arrived at the following well known result: *If the equation of compound interest is solved for the time by linear interpolation, the result is precisely the time at the end of which the practical compound amount on the given principal P equals the given amount A.*

4. *Determination of a final partial payment.* In the equation

$$A = Ra_{\overline{n}|i}, \tag{6}$$

i is the interest rate per conversion period, and A is the present value of an annuity of R payable at the end of each period for n periods. This description of equation (6) has meaning only when n is an integer, although the algebraic expression for $a_{\overline{n}|}$ is defined for all values of n. The question arises as to the interpretation which can be given to a non-integral solution for n in equation (6) if A, R, and i are given. No attempt will be made to interpret a value of n obtained by solving this exponential equation by use of logarithms. We shall suppose that n is obtained by use of linear interpolation in a table of values of $a_{\overline{k}|}$ for integral values of k. By use of equation (3) with $f(n) = Ra_{\overline{n}|i}$, we find that such a solution of equation (6) assumes that, when $n = k+h$, A is given by

$$A = Ra_{\overline{k}|i} + hR(a_{\overline{k+1}|i} - a_{\overline{k}|i}). \tag{7}$$

By use of the geometric progressions for $a_{\overline{k}|}$ and $a_{\overline{k+1}|}$, from equation (7) we obtain

$$A = Ra_{\overline{k}|i} + hR(1+i)^{-(k+1)}. \tag{8}$$

Thus, when we solve equation (6) for n by interpolation, we are actually solving equation (8) for two unknowns k and h, where k is an integer and $0<h<1$. We note that the right member of equation (8) represents the present value of an annuity of R per interest period for k periods, plus the present value of a smaller payment hR due at the end of $(k+1)$ periods.

Now, consider a debt whose present value is A, which is to be discharged, principal and interest at the rate i per conversion period included, by payments of R at the end of each period. For given A, R, and i, it would be unusual if an integral number of payments R exactly sufficed to extinguish the debt A. Hence, usually, a final smaller payment $M<R$ will be necessary one interest period after the last full payment R. If A, R, and i are known, we desire to find the number, k, of regular payments R and the concluding payment M due at the end of $(k+1)$ interest periods. If we let $(M \div R) = h$, or $M = hR$, it is verified that the equation which the unknowns h and k must satisfy is identical with equation (8). Thus, in view of the discussion leading to equation (8) we can state the following conclusion:

To find the number of payments of R per interest period and the final partial payment M which will discharge a specified debt A, principal and interest included, first solve $A = Ra_{\overline{n}|i}$ *for n by linear interpolation, obtaining* $n = k+h$, *where k is an integer and* $0<h<1$. *Then, the debt requires k payments of R each, and a final payment* $M = hR$ *due at the end of* $(k+1)$ *interest periods.*

Illustration: Suppose a debt of \$8800, with interest at the rate 5%, compounded semi-annually, is to be discharged by payments of \$1200 at the end of each 6 months. We write

$$8800 = 1200a_{\overline{n}|.025},$$

whose solution, obtained by interpolation, is $n=8.20381$. Hence, the debt requires eight payments of \$1200 each, and a final payment

$$M = (.20381)(1200) = \$244.57$$

at the end of $4\frac{1}{2}$ years. The reader can easily verify that this result is accurate to the nearest cent, by determining M by use of an equation of value.[1]

5. *A problem relating to the amount of an annuity.* Consider the equation

$$S = Rs_{\overline{n}|i}, \tag{9}$$

when S, R and i are given. Suppose that we obtain $n=k+h$ on solving for n by linear interpolation in a table of values of $s_{\overline{k}|}$ for integral values of k. By use of equation (3) with $f(n)=Rs_{\overline{n}|i}$, we find that the interpolated value of n is the exact solution of equation (9) under the assumption that, when n is of the form $n=k+h$, S is given by

$$S = Rs_{\overline{k}|i} + hR(1+i)^k. \tag{10}$$

The right member of equation (10) represents the amount of an annuity of R per interest period for k periods, plus the compound amount at the end of the term of the annuity if the additional payment hR is invested at the beginning of the term. If we solve equation (9) for n by interpolation, we are actually solving equation (10) for the unknowns (h, k), where k is an integer and $0<h<1$, and our result $n=k+h$ can be interpreted by reference to the preceding sentence.

Illustration: If the equation $10{,}000=500\,s_{\overline{u}|.025}$ is solved for n by linear interpolation, the result is $n=14.20272$. We find that $(.20272)(500)=\$101.36$. Hence, if interest is at the rate 5% per conversion period, an initial deposit of \$101.36, plus payments of \$500 at the end of each interest period, will accumulate exactly \$10,000 by the end of 14 periods.

II. On the Limit of the Ratio of an Arc to its Chord

By H. A. Simmons, Northwestern University

1. *Introduction.* Let $y=f(x)$ be a function of one of the following types[2]: (A) one-valued, with a continuous derivative at each point of a closed interval from $x=0$ to $x=k$ (where k is a finite constant conveniently assumed to be greater than 0) except at the point for which $x=k$, at which point $f(x)$ becomes infinite, e.g., $\lim_{x\to k} f(x)=\pm\infty$); (B) one-valued, with a continuous derivative for every $x\geqq 0$, and such that $\lim_{x\to\infty} f(x)=k_1$, where k_1 is any constant; (C)

[1] See W. L. Hart, *Mathematics of Investment*, D. C. Heath and Co., p. 83.

[2] The following examples may help one to see what are the types of functions (A), (B), (C) which we consider: the curve $y=\sec x$ comes under type (A); $y=1/(x+1)$ comes under type (B); $y=e^x$ comes under type (C); $y=\sin x$, which neither approaches a constant nor becomes infinite as $x\to\infty$, is excluded from consideration; $r=a^\theta$, which when expressed in the form $y=f(x)$ is multiple-valued, is also excluded.

one-valued, with a continuous derivative for every $x \geqq 0$, and such that $\lim_{x\to\infty} f(x) = \pm\infty$.

Consider the ratio of the length s (greater than zero) of an arc of the curve $y=f(x)$ between the point $P[0, f(0)=a]$, where a is finite, by our hypotheses on $f(x)$, and the point $Q[x, f(x)]$ to the length c (greater than zero) of the chord PQ. It is well known that the limit of this ratio as $c\to 0$ is 1. In this paper, we wish to prove that the limit of this ratio as $c\to\infty$ is also 1.

2. *Cases to consider.* Since $y=f(x)$ has a continuous derivative on its interval except possibly at a right hand end-point [See (A), (B), (C) above], the arc length s from P to Q, where Q is not such an end-point, can be represented by a customary formula of calculus; hence

$$\frac{s}{c} = \left[\int_0^x (1+f'^2)^{1/2}dx\right] \Big/ [x^2 + (f-a)^2]^{1/2}$$

an expression involving three variables x, $f=f(x)$, and $f'=f'(x)$. Reference to (A), (B), (C) above will now show that we need only to consider the following cases[2]:

(I) $\lim_{x\to\infty} f = \pm\infty$, $\lim_{x\to\infty} f' = \pm\infty$; (II) $\lim_{x\to k} f = \pm\infty$, $\lim_{x\to k} f' = \pm\infty$;

(III) $\lim_{x\to\infty} f = \pm\infty$, $\lim_{x\to\infty} f' = k_1$; (IV) $\lim_{x\to\infty} f = k_1$, $\lim_{x\to\infty} f' = k_2$;

where k is a positive constant and k_1, k_2 are arbitrary constants.

3. *Discussion of cases.* We apply customary rules of calculus for evaluating indeterminate forms, and in obtaining (2) from (1) we use a formula from the theory of definite integrals, namely,

$$\frac{d}{dx}\int_0^x \phi(t)dt = \phi(x).$$

(I). Differentiating separately the numerator and denominator of (1), we find

$$\lim\frac{s}{c} = \lim\frac{(1+f'^2)^{1/2}[x^2+(f-a)^2]^{1/2}}{x+(f-a)f'}. \tag{2}$$

Dividing numerator and denominator of the second quotient in (2) by $(f-a)f'$, and observing that $\lim[x/(f-a)] = \lim(1/f') = 0$, we find

$$\lim\frac{s}{c} = \lim\frac{[(f')^{-2}+1]^{1/2}[x^2(f-a)^{-2}+1]^{1/2}}{x(f-a)^{-1}(f')^{-1}+1} = 1.$$

(II). Proceed as in (I).

[2] One might think it also necessary to consider the cases: (i) $\lim_{x\to\infty} f = k_1$, $\lim_{x\to\infty} f' = \pm\infty$; (ii) $\lim_{x\to k} f = \pm\infty$, $\lim_{x\to k} f' = k_1$; (iii) $\lim_{x\to k} f = \pm k_1$, $\lim_{x\to k} f' = \pm\infty$; where k is a positive constant and k_1 is an arbitrary constant. But it can be shown by an application of the mean value formula $f(b)-f(a) = (b-a)f'(x_1)$, $a<x_1<b$, that cases (i) and (ii) are impossible; while the case (iii) is obviously inconsistent with the hypothesis that $c\to\infty$ (See §1).

(III). Divide numerator and denominator of the second quotient of (2) by x and use the fact that $\lim [(f-a)/x] = \lim f' = k$; the result follows immediately.

(IV). The hypotheses that x approaches ∞ and $\lim f = k_1$ imply that $\lim f' = 0$ as can be shown by an application of the mean value formula stated in the footnote No. 2. Hence division in the second quotient of (2) by x leads to the result.

4. *Consequences.* The functions $y=f(x)$ for which $\lim (s/c)$ as $c \to \infty$ is 1 are bounded and continuous on a closed interval from $x=0$, to $x=k$, or from $x=0$ to $x=\infty$, the meaning of ∞ as a right hand end-point being clear from § 1. These functions therefore attain a maximum value (and the minimum value 1) on their intervals.

This result is surprising in the cases of numerous functions. For example, one would scarcely believe at first thought that s/c attains a maximum in the case of the parabolic arc $y=2(ax)^{1/2}$.

III. A Note on a Simple Theorem in Relative Velocity

By A. F. Stevenson, University of Toronto

If a closed circuit of any shape is marked out in a river flowing with uniform velocity, the circuit being fixed with respect to the bank, then a boat, whose speed in still water is constant, takes the same time to journey round the circuit, in whichever sense the circulation is completed.

This result would be suspected in advance and is easily proved; but inasmuch as no mention of it appears to be made in the text-books, it may perhaps be worth while to reproduce a proof. We can prove the result, somewhat more generally, when the flow of the river is not uniform, provided only that it is steady, and that the velocity of flow is the same (in sign and magnitude) at all points of the circuit which are at the same distance downstream.

Let V, v denote, respectively, the constant velocity of the boat in still water, and the velocity of flow at any point of the circuit. (We must, of course, suppose $V > |v|$ everywhere.) Also let ds denote an element of the circuit at a point whose downstream distance from some fixed origin is x, and let ds make an angle θ with the x-axis.

Then for the velocities of the boat relative to the circuit when traversing ds in the two directions, we easily find

$$v_1 = \sqrt{(V^2 - v^2 \sin^2\theta)} - v\cos\theta;\ v_2 = \sqrt{(V^2 - v^2\sin^2\theta)} + v\cos\theta).$$

Hence the *difference* in the times taken to traverse ds in the two directions is

$$ds\left(\frac{1}{v_1} - \frac{1}{v_2}\right) = \frac{2v\,ds\cos\theta}{V^2 - v^2} = \frac{2v}{V^2 - v^2}dx,$$

where dx is the projection of ds on the x-axis. The difference in times for the whole circuit is therefore $\int 2v(V^2-v^2)^{-1}dx$, taken round the circuit. But from the

hypotheses made as to v, it follows that the integrand is a single-valued function of x only; the value of the integral is therefore zero, and the theorem is proved.

We can further suppose that the direction of flow is also variable; but in this case the direction of flow and the circuit must both satisfy restrictive conditions of symmetry. In the particular case where the direction and velocity of flow are the same everywhere, we see from the above that the difference in times for an *unclosed* circuit is proportional to the downstream distance between the end points; and, whether v is constant or not, depends only on this distance, the shape of the circuit being immaterial.

IV. Recurrence Formulae Involving Bessel Functions of the First Kind

By W. O. PENNELL, St. Louis, Mo.

The object of this note is to call attention to the following formulae involving Bessel functions:

$$(1)\ (2n+1)\int_0^\theta x^n\epsilon^{\alpha x}J_n dx = \left[x^{n+1}\epsilon^{\alpha x}(J_n - \alpha J_{n+1})\right]_0^\theta + (\alpha^2+1)\int_0^\theta x^{n+1}\epsilon^{\alpha x}J_{n+1}dx,$$

$$(2)\ (2n+1)\int_0^x x^n\epsilon^{\pm ix}J_n dx = x^{n+1}\epsilon^{\pm ix}(J_n \mp iJ_{n+1}),$$

$$(3)\ (2n+1)\int_0^x x^n \cos x\, J_n dx = x^{n+1}\cos x\, J_n + x^{n+1}\sin x\, J_{n+1},$$

$$(4)\ (2n+1)\int_0^x x^n \sin x\, J_n dx = x^{n+1}\sin x\, J_n - x^{n+1}\cos x\, J_{n+1},$$

$$(5)^*\ (2n+1)\int_0^x x^n\epsilon^{\mp x}I_n dx = x^{n+1}\epsilon^{\mp x}(I_n \pm I_{n+1}),$$

$$(6)\ (2n+1)\int_0^x x^n \cosh xI_n dx = x^{n+1}\cosh xI_n - x^{n+1}\sinh xI_{n+1},$$

$$(7)\ (2n+1)\int_0^x x^n \sinh xI_n dx = x^{n+1}\sinh xI_n - x^{n+1}\cosh xI_{n+1}.$$

Formulae (2) to (7) are special cases of (1). In all the formulae $n > -1/2$.

Some search has been made through the standard works on Bessel functions without finding any mention of these relations and they are published with the hope that they may be of some service.

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* The notation $I_n(x)$ is used to denote $i^{-n}J_n(ix)$.

The proof of (1) is as follows: Two integrations by parts give:

$$(n+1)\int x^n e^{\alpha x} J_n(x)dx = x^{n+1}e^{\alpha x}J_n - \alpha\int e^{\alpha x}x^{n+1}J_n dx - \int e^{\alpha x}x^{n+1}J_n' dx$$

$$= x^{n+1}e^{\alpha x}[J_n - \alpha J_{n+1}] + \int e^{\alpha x}x^{n+1}[\alpha^2 J_{n+1} - J_n']dx.$$

Upon replacing J_n' by $J_n' = nx^{-1}J_n - J_{n+1}$, and transferring the resulting term in J_n to the left member of the equation, we get the recursion formula (1). (2) is obtained from (1) by giving α the values $\pm i$. (3) and (4) are obtained by taking respectively half the sum and half the difference of the two values of (2). (5) is obtained from (1) by giving α the values $\pm i$ and making θ imaginary. (6) and (7) are obtained by taking respectively half the sum and half the difference of the two values of (5).

When $2n$ is not an integer the origin is a singular point of $x^n\epsilon^{\alpha x}J_n(x)$ and in this case it would have to be understood that the path of integration is either a radial line from 0 to θ or its equivalent.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Hunter College of the City of New York.

All books for review should be sent directly to the editor of this department and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Currier, C. H. and *Watson, E.* A Course in General Mathematics. New York, Macmillan, 1929. x+414 pages.

Prasad, Ganesh. Six Lectures on Recent Researches in the Theory of Fourier Series. Published by the University of Calcutta, 1928. xiv+140 pages.

" . . . in as compact and simple a form as possible, the present state of the convergence problem in the theory of Fourier series."

Hessenberg, Gerhard. Vorlesungen über Darstellende Geometrie, herausgegeben von E. Salkowski. Leipzig, Akademische Verlagsgesellschaft M. B. H. xiv+274 pages.

Schwam, W. Elementare Geometrie. Erstes Band, Die Ebene. Leipzig, Akademische Verlagsgesellschaft M. B. H. xviii+402 pages.

Two new volumes in the Hilb series "Mathematik und ihre Anwendungen in Monographien und Lehrbüchen."

Springer, Julius. Quellen und Studien zur Geschichte der Mathematik. Herausgegeben von O. Mengebauer, J. Stenzel, O. Toplitz. Abteilung B: Studien, Band 1, Heft 1, Berlin, 27 März, 1929.

Contents: Das Verhaltius von Mathematik und Ideenlehre bei Plato, von O. Toeplitz. Zur Theorie des Logos bei Aristoteles. Zur Geschichte der Babylonishen Mathematik, von O. Neugebauer. Über die Geometrie des Kreises in Babylonien von O. Neugebauer und W. Struve. Platos Einfluss auf die Bildung der Mathematischen Methode, von J. Johnsen.

Oglesby, E. J. and *Cooley, H. R.* Plane Trigonometry. New York, Prentice Hall, 1929. x+154+76 pages, with tables. $1.60.

Nevalinna, R. Le Theoreme de Picard-Borel et la theorie des fonctions meromorphes. Paris, Gauthier-Villars, 1929. viii+176 pages.

REVIEWS

Readers who are interested in the reviewing of books are invited to write to the editor of this department indicating particular books which they would like to review or the kinds of books in which they would be interested.

Gino Loria, Professore nell'università di Genova, *Storia delle Matematiche, Vol. I, Antichità—Medio Evo—Rinascimento. Con numerose figure nel testo.* Torino, Società Tipografico-Editrice Nazionale, 1929, 497 pages.

The appearance of a new history of mathematics by a writer as widely and favorably known as Gino Loria is to be hailed with satisfaction. Loria's aim is not to present a severely technical account of mathematical development, but to make the story readable by a popular and attractive style of exposition. Combining readability and the requirements of scientific accuracy is an undertaking in which it would be difficult to surpass the distinguished historian from Genoa.

In fifteen chapters of this, the first volume, the story is told, beginning with the number systems, the intuitive geometry and astronomy of Babylonia and Egypt, and ending with the syncopated algebras of Widman and Pacioli. Somewhat unusual chapter headings are "The Chinese Enigma," referring to the difficulty of historical research in Chinese mathematics and the variety of judgments on the achievements of Cathay, and "Geometry in the service of painting" as seen in Leonardo da Vinci and Albrecht Dürer. Of questionable propriety is "S. P. Q. R.," signifying *Senatus Populusque Romanus*, and used as the heading of the chapter on Roman mathematics.

The account on Babylonian astronomy unfortunately omits reference to the greatest achievement of observational astronomy of antiquity, namely, the discovery of the precession of the equinoxes. In histories of astronomy this great discovery has been ascribed to the Greek astronomer Hipparchus. Loria so ascribes it. But it is now proved conclusively that a few centuries before Hipparchus, the Babylonian astronomer Cidenas[1] on the banks of the Euphrates had reached this result. As regards the Egyptians, recent research is revealing more and more clearly that the early dwellers of the Nile had proceeded much further in mathematics than is indicated by Greek authors. Of this fact the Rhind papyrus was the first distinct indication. Loria refers to an Egyptian papyrus now being studied in Russia, but not yet published in full. It contains a calculation of the surface-area of a hemisphere, made 1500 years before Archimedes! It is worthy of remark that Loria overlooked an equally startling result in the Moscow Egyptian papyrus[2] containing the computation of the

[1] See Paul Schnabel in *Zeitschrift für Assyriologie*, N. S., vol. 3 (1926), pp. 1–60.

[2] *Ancient Egypt*, 1917, p. 100.

volume of the frustum of a square pyramid, at a period of 1500 or 2000 B.C.

Loria's treatment of Greek mathematics is in general of high quality. However, we were disappointed in his discussion of Zeno's arguments against motion. In the first place he fails to indicate the perennial interest in this topic as revealed even by recent writers, for instance Bertrand Russell and Henri Bergson. Nor is there a reference to Paul Tannery's brilliant interpretation of Zeno's arguments. In the second place, Loria does not make plain how profoundly Zeno of Elea influenced Greek science by indirectly inducing philosophers to advance the atomic theory, the "atom" (indivisible) being created to afford what seemed to be a safe escape from the subtle grip of Zeno's logic. It seems true also that Zeno's arguments were partly responsible for the banishment from Euclidean geometry of the notion of the fixed infinitesimal and the infinite. It is pleasing to find in Loria a heading "Arithmetical recreations of the Greeks." In fact, mathematical recreations are almost as old as mathematics itself. Witness in the Rhind papyrus Ahmes' problem, leading to a geometrical progression, and interpreted to signify: Seven persons have each seven cats, each cat eats seven mice, each mouse eats seven ears of barley, from each ear seven measures of corn may grow. How many persons, cats, mice, ears and measures altogether?

Loria's book possesses qualities which are certain to stimulate general interest in the history of mathematics.

FLORIAN CAJORI

Freshman Algebra. By James Byrnie Shaw. Thomas Y. Crowell Co., New York, 1929. 137+xi pages.

I. Shaw's *Freshman Algebra* is a text to be used primarily by women's divisions in college algebra.

The book is not intended to be used in the usual courses in algebra where the subject is applied in as many ways as time permits. It is the author's purpose to present algebra devoid of applications, stressing the beautiful and the aesthetic. Poetry, rather than utility, furnishes the background for the course.

Many topics are treated in comparatively few pages, each in a necessarily superficial manner. The subjects include integers, equations, determinants, irrationals, identities, invariants, and hypernumbers. It is to be remembered, however, that the book is not a mathematics text in the ordinary sense; it is rather a mathematical companion for a course of study in the purpose, beauty, and mission of mathematics, or, more specifically, algebra. After such a course it seems reasonable that the students might have clearer conceptions of what are the real value and place of algebra than most students acquire from the present courses.

"The book is not intended to contain everything that is said or done in the class. The author assumes that the class is in charge of a teacher who is interested enough to bring in any extra material. . . ." The contents are chiefly explanatory, describing briefly the various topics, the fundamental operations

with each, and the type of work that can be done with each. Solution of the usual types of problems is minimized, and there are very few problems for illustration and solution by students. Although the course of study is intended to be based on literature and fine arts there seem to be very few references to these phases; this material being left to the instructor to furnish. It seems as if more specific references to poems and other literature would prove a helpful addition in guiding an instructor toward making the best use of the text and conducting a better course.

The book is novel and will be received with welcome by many who have courses where such a text can be used. The explanations are especially clear, are presented in a conversational manner, and should prove appealing to students.

E. C. WARREN

II. In these days of statistical investigation into theories of education, with resultant revolutions in our college and secondary school curricula, mathematics almost more than any other single subject has received a great deal of attention. Old theories of transference of values are being discarded despite the repeated warnings of certain groups of educators steeped in the theory of discipline in mathematics and the classics. No one of course attempts to deny the utilitarian place of mathematics in the education and training of the engineer and scientist, for in all studies of such curricula, investigators feel the need for more time and more thorough training in subject matter and methods of mathematical development. But it has come to be a different story for the so called "liberal arts" group, to whom the mathematics requirement is just one more hurdle to leap in the path of credit gathering for a degree. The student in this group will probably never have occasion to use any of the actual subject matter of his freshman year, aside from some incidental work in arithmetic, of which, in most cases, he is woefully ignorant. The solution in an increasing number of institutions is to strike mathematics entirely out of the curriculum, or allow as an elective some other course.

Mathematics, however, has too extensive a background in the development of education to be so lightly discarded simply because the student does not like the usual doses he receives of it, or feels he has had enough "discipline" to enable him to tackle something more attractive. The mathematician may feel that his standing and the standing of his field in the intellectual world is endangered if we allow the ancient subject to be replaced in our curricula. But perhaps the teacher of mathematics, himself, is somewhat to blame for the whole trend of development in the liberal arts courses. Professor Shaw's book seems to the reviewer to be a big step, if not too radical, in the direction towards a solution of the difficulties sketched above. It may be dangerous, for in the hands of the inexperienced, unsympathetic, mathematical drill master it would be more a farce than some of our ordinary courses. In the hands of a genuine master of his field with sympathetic guidance, extensive mathematical training, keen interest in arts, literature, and philosophy, knowledge of the history of

developments in modern mathematics, and above all an unbounded enthusiasm for the subject himself, it will form the basis for a course of utmost value and interest for the student. I will not maintain that he will solve equations better or expand binomials more accurately, or manipulate logarithms more quickly than the student who has had the ordinary drill and routine of college algebra, although it is certainly possible. He will, however, see something of the beauty of mathematics, have some knowledge of its contribution to the advance of civilization, know something of its aesthetic qualities, which any person who aspires to be a cultured individual should feel when viewing such a sublime creation of the intellect, and the conviction that mathematics is a living, growing, vital part of the intellectual life of the world.

The author is convinced that mathematics would be more properly classed under the "fine arts." Its relation to poetry is constantly stressed with numerous examples and references. Its symmetry, design, rhythm, harmony, unity, creativeness are some of the aesthetic properties analogous to the same properties of other fields of art that are frequently pointed out and developed. Nor is the book entirely a discourse *about* mathematics, for a wealth of subject matter is presented, some of which is given in a manner wholly out of the ordinary, showing the author's individuality in methods. The approach is generally from an intuitive standpoint, encouraging the student to use his imagination to arrive at the general conclusions before they are actually stated by the author. The reviewer feels that unless the teacher in planning the course is careful to make his own background as broad as possible, the student will get just as naïve notions about mathematics as he would in an ordinary course and more likely extremely fantastic conceptions. Experience and personal limitations ought to guide the instructor as to how far he should go with the more unusual topics introduced.

The text is not long and might otherwise pass as a well written set of notes on lectures delivered by the author. The style is somewhat informal and discursive making the contact between the author and the student more potent. The first four pages give a concise account of the place of algebra in history with several references for further reading. This closes with a list of the divisions of the work, namely: arithmetic, equations, identities, invariants, operators and hypernumbers. Chapter II gives a number of theorems on divisibility of integers as examples of number theory, with the binomial coefficients as the goal and end of the chapter.

Chapter III is on elimination, bringing in naturally determinants and their properties and, as an unusual topic, Sylvester's dialytic method. Chapter IV is a discussion of the rational domain and some remarks on the connection between mathematics and other fields of creative thought. Chapter V, which is the most extensive, deals with equations in one unknown, with presentations in an individualistic manner of such topics as rational roots, Horner's method, use of continued fractions in approximating roots, symmetric functions, Galois resolvent equations, and isolation of roots by Sturm's functions. Chapter VI is

on irrational numbers and laws of exponents with logarithms introduced incidentally. Chapter VII is on identities, sums of finite series, method of differences, partial fractions (though not so named), arrangements and probabilities. Chapters VIII and IX give short introductions to the notions of groups, invariants, and quaternions, closing with a brief mention of the direction of continuation of the various topics considered.

A half dozen or so misprints were noticed. Most of them are quite obvious and can be corrected by the student. Large type was used throughout in the make-up of the book, making it mechanically easily readable. Some minor departures from usual printing were used, scarcely detracting or adding to the book's worth.

The personality of the instructor, master of his subject, a refined, cultured individual acquainted with philosophy and art and filled with a conviction of the place of mathematics in the make-up of a well educated person, must dominate such a course for which Professor Shaw's book would form a basis. It is perhaps most of all a sketch of some of the author's methods of treatment and an indication of how the individual instructor may vary the work to the tastes and make-up of the class. The reviewer believes that it is at least a very potent answer to the problem of the liberal arts student although inadequate for use by the inexperienced instructor with insufficient mathematical training. It should arouse perplexing questions in the mind of the student demanding some kind of intelligent exposition.

A competent, open minded teacher seeking a modus vivendi for the liberal arts freshman in mathematics might do well to conduct an experiment of value to himself if not to others. Let him take two classes of equal ability and previous training so far as can be readily determined and let him give to one class as a control a course based upon the conventional text book with the conventional material and to the other a course based on work such as Professor Shaw has sketched in his *Freshman Algebra*. Then let him draw his own conclusions from the results as he sees them, not measured by some examination of manipulative skill. Let him expect different results from the essentially different aims just as we expect different results from a student in a course in music appreciation and a student of piano technique.

CLYDE M. HUBER

First Course in the Differential and Integral Calculus. By. W. B. Ford. Henry Holt and Co., New York, 1928. 372 pages. $3.00.

This introduction to the calculus by Professor Ford of the University of Michigan follows, to quote from the preface, "in large measure the lines of development already customary in American texts on the calculus. . . . There are 2005 examples and exercises. Of these, 179 are worked in full and serve as illustrative material . . . . Of the remainder, 89 are accompanied by both the answer and a hint as to method of solution, 1132 are accompanied by the answer

only. . . . As to the proofs, it is well known that it is impossible in an ordinary first course in the calculus to furnish strictly vigorous proofs at all points. . . . The best one can do . . . is to give what may be termed a 'first approximation' to a real proof. This, at least, is all that has been attempted in the present text, the student being told plainly, however, that such is the case . . . . Special attention has been given to the applications of the calculus to physics. . . . A brief treatment of the method of least squares has been given in the Appendix. . . . Tables at the end of the book include not only a considerable number of the more important standard integrals, but also trigonometric formulas and tables showing the sine, cosine, and tangent, correct to four decimal places, of all angles, expressed in either degrees or radians, at intervals of 10′ from 0° to 90°. . . . A table has been added also showing the squares, cubes, square roots, cube roots, and reciprocals of all integers from 1 to 100."

The scope and plan of the book appear from these quotations. The purpose of the author has been in our opinion fairly well carried out. Certainly any student who worked through the text would have a good knowledge of the elements of the calculus. The appearance of the book is good, with a large, clear page.

The book contains too much for a first course, if by a first course is meant a one-year course. There is a great lack of problems calling for numerical work. The principles and methods of the calculus are best brought home to beginners by continual emphasis on numerical examples. In this connection we note that the book contains no table of natural logarithms, probably a much more useful table in a calculus than a table of common logarithms; the trigonometric tables would be more useful if the functions were given for every hundredth or thousandth of a radian rather than for every ten minutes. The fundamentally important subject of formal integration seems to us to be treated rather lightly: the methods and devices over which the beginner stumbles and struggles appear largely as hints for the solution of problems and do not receive the systematic explanation and discussion which seem to us their due.

The following misprints were noted: on page 30, example 1; the answer should have a factor 2 in the denominator. On page 83, the graph for example 2 is inverted so as to represent $y=x/(x^2-1)$ instead of $y=x/(1-x^2)$. On page 87, exercise 4 is incorrect; the locus, incidentally, is composite. On page 149, fifth line from bottom, for x^1 read x_1. On page 161, exercise 3c; the answer should be half the given value. On page 180, exercise 3a; the value of e is given as 2.7182 . . . instead of 2.7183.

J. K. WHITTEMORE

Plane Trigonometry. By Carl A. Garabedian and Jean Winston. McGraw-Hill Book Co., 1929. xviii+306 pages.

In looking back over this text after spending a good deal of time reading its eight chapters, I am struck with three things: the very unique style of presen-

tation, the completeness of the explanations, and something more vague which might be called the "atmosphere" of the book.

The style, called by the authors "the syllabus mode of presentation," is like a very complete set of notes; and if a student should master this style, it would certainly improve his ability to take intelligible notes in a mathematics lecture. It catches the attention and gives the impression that everything said is important, and one can easily see just what is said in any particular article. Here is a sample:

"36. *Application of trigonometry to the solution of triangles.*—Before attempting problems in plane triangles, we first *note that*

(a) Certain important facts concerning triangles should be recalled from plane geometry (Art. 13).

(b) To *solve* a triangle is to find numerical values for *all parts not given.* There are two possible solutions:

(1) the *graphical solution*, whose importance should not be underestimated; and

(2) the *trigonometric solution*, which we shall discuss fully below.

(c) A triangle can be solved provided we have *given three parts, at least one of which is a side* [Art. 13, (7)]. (Is the solution always unique?)

(d) In labelling the parts," etc.

The explanations and illustrative examples are unusually complete, the computations are put up in attractive form, and the graphs are beautiful with orange rulings and black curves. This careful work is commendable, and has a real influence on the attitude of the student and his working habits. I think, however, that the authors have overdone it in some places; for instance they devote sixteen pages to "Accuracy of measurements, tables, and computed results"; and the explanations are almost confusingly minute as to how to round off a number when it has too many digits, and the abbreviated methods of multiplication and division. Again, the treatment of "Graphs of the functions" begins with an article on graphing in general and plots a few conic sections, and ends up with compound curves obtained by adding several sine or cosine curves. This takes up about forty pages. It seems to me that if some of these things could be shortened and a set of tables put in their place the book would be more teachable. The authors meet this objection in the preface by saying that since we are so often called upon to give a short course the student needs a complete text all the more, especially if he is going to keep his text and use it for reference later.

In this connection, I should like to voice a thought that has been in my mind for some time. Correspondence courses are becoming increasingly popular, and here the student must indeed depend upon the text. Should we develop a special type of text for these courses? The text we are reviewing would make a very good one for a correspondence course.

The "atmosphere" of the text shows clearly that the authors love their subject and feel that no apology is needed for expecting the students to find an

interest in it. I hardly know how this idea is conveyed throughout the text unless it be by the questions that are constantly raised for discussion. The reader will notice one tucked in under a sub-topic in the extract quoted above. They never pass quietly by a difficult point like the tangent of ninety degrees, but always raise the question even though many questions are raised only to be postponed. In the introduction, after saying that trigonometry is a necessary stepping stone to higher mathematics, the authors say that it "possesses a certain beauty and significance characteristic of an abstract mathematical theory—a beauty comparable in its purity and simplicity to that of a fugue of Bach or a symphony of Beethoven." In a short conclusion, (an unusual feature), they advise the student to review the course for the sake of perspective, and tell him that he is only at the beginning of a large and interesting subject. The text is in all respects consonant with an observable tendency to let the pendulum swing back from the utilitarian toward the cultural.

DAVID F. BARROW

Short Course in Spherical Trigonometry. By Pauline Sperry. Johnson Publishing Company, vi+57 pages.

Short as is this spherical trigonometry, containing only fifty pages, we find in these comparatively few pages a geometrical setting, development of the theoretical spherical trigonometry, applications, and a brief historical sketch. These are set forth in a remarkably complete presentation of the subject, showing careful work at every step. The technique of the composition is good. One is immediately impressed with the attractive page which has good arrangement, clear print, and unusually fine drawings.

The section referring to spherical geometry is none too explicit for review on the part of the student who has studied that subject, yet is sufficiently detailed and logically presented to guide carefully through that ground the student who meets this material for the first time. Such a student would acquire, from the chapters on the sphere, a vivid understanding of the ideas prerequisite to work on the trigonometry of the spherical triangle. The clear-cut definitions and the arrangement of the theory would produce an enquiring state of mind. In this portion of the text the author has presented especially well chosen sets of thought-provoking exercises.

Condensation of theory by means of the principle of reciprocity, complete treatment of the ambiguous case, and stimulating exercises are prominent features of this small book. In the computational work good sets of exercises are given, completely covering the theory presented. The problems for application of this part of the work are both varied and refreshingly modern. The emphasis laid on checking results is very cheering.

In calling attention to many connections between plane and spherical trigonometry, the author enriches the work for the student of lesser initiative. In this fact and in other ways the book brings out what we so often optimistically

hope for, and so seldom find—a treatment of a portion of mathematics in its relation to other connecting material, not as a detached subject to be pigeonholed and separated from its close relatives.

MARY E. WELLS

Studies in the History of Statistical Method—With Special Reference to Certain Education Problems. By Helen M. Walker. Baltimore, Williams and Wilkins Company, 1929. viii+229 pages.

The study of mathematical statistics is just now attracting many students with a foundation of sound mathematical training, but without experience in any field of statistics. In even greater numbers it is gaining recruits among biologists, psychologists, economists, anthropologists and others who hope to find in mathematical statistics the solutions of some of their own special problems. To a person who is approaching this study the first great difficulty is to define the field. This is undoubtedly true of any subject but it is especially true of statistics. This results in part from its topical nature, the methods and purposes of the various topics being quite different. In part, too, it results from the important contributions made by men working in different lines, dealing with different material and often ignorant, as Dr. Walker points out, of results and terminology already in use in other fields. These forces have not only created a field which is difficult to define; they have brought about a great confusion and duplication of formulas, terminology, and notation. It is doubtless among the functions of the mathematicians who are teaching or studying statistics to coördinate the work already done in different portions of the field, to simplify and standardize notation and terminology, and to unify and define the subject. Certainly one of the most satisfactory ways of forming a clear and unified picture of the science as it now stands is by constructing a background of its historical development. For this reason I predict a warm welcome for Dr. Walker's "Studies in the History of Statistical Method."

The title of the book indicates its topical and incomplete character. The two longest chapters are those on the normal curve and correlation. The first of these is supplemented by shorter chapters on moments and percentiles. Written from the point of view of one especially interested in problems of educational psychology, there is a chapter on Spearman's "Theory of Two Factors" and another on the history of the teaching of statistics in American universities. Those who are interested in the more general aspects of statistical study will perhaps share my regret that these two chapters were included at the sacrifice of material on curve fitting, analysis of time series, graduation, least squares, sampling or some of the many other important topics of general interest which were omitted. We may hope that these will be treated in later studies.

The concluding chapter of the book is an alphabetical list of certain technical terms used in statistics together with the name of the person who introduced each term and references to its first publication. This list unfortunately does not

include symbols, and it is most complete in the domain of educational statistics. As the author suggests, this material should be of service to an investigator who proposes a new term or formula by acquainting him with the sources of terms already in use. It would have been still more useful, even to the student of educational statistics, if the terms of other branches of statistics had been included. For duplication of terms results precisely when a writer introduces a new expression for which an equivalent already exists in a part of the field with which he is not familiar. It would be ungrateful, indeed, however, not to appreciate this chapter as the nucleus of a historical dictionary of statistical terms.

Similar to this dictionary are the special bibliographies. The chapter on the normal curve ends with a bibliography of memoirs on the probable errors of frequency constants arranged according to the year of publication. There is another list of memoirs relating to measures of correlation other than the Pearson product-moment, likewise arranged by years, and with annotations amounting to abstracts of the papers. Another annotated bibliography is given for the theory of two factors. Unquestionably these special bibliographies will do more than anything else to make this book valuable as a work of reference. As is true of so many features of the book they suggest a method which, being topical itself, can be extended to topics not considered here. This goes far to answer any criticism that may be made on the score of incompleteness.

The topical method of the bibliographies is largely carried out in the text. Thus in the chapter on the normal curve, after discussions of the origin in the theory of probability and the establishment of the normal curve, we find histories of the application of the law of error to social phenomena, of the use of different units for the abscissa of the curve, of the development of tables of the error function, and of the search for probable errors of frequency constants. This method is sound, I think, and the author has done well to cover portions of the field intensively rather than to attempt a less thorough survey of a wider territory. The general bibliography should be useful, but includes only works actually consulted by the author. This gives impressive evidence of the author's background of reading, but causes some curious omissions.

The author has a keen perception of what is important. It is natural that some of her decisions should surprise us, as, just as instances, her entirely unqualified tribute to the influence of Quetelet, her judgment of the work of Jacob Bernoulli and her negative remark concerning the nature of a Type B Gram-Charlier series.

The style is easy and lively. The book is not intended as a text-book in statistical theory, but is addressed mainly to readers who are somewhat familiar with the subject. It is good to read and good to refer to. I wish that I had had it at hand during the past year while teaching a course in the mathematical theory of statistics. From now on it should be an important resource for teachers and investigators.

RAYMOND W. BRINK

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEM FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3385. *Proposed by J. Rosenbaum, Milford, Conn.*

In any pentagon, $A_1A_2 \cdots A_5$, P_1, P_2, $\cdots$, P_5 are the midpoints of the sides A_1A_2, A_2A_3, $\cdots$, A_5A_1. The points M_1, M_2, $\cdots$, M_5 are the midpoints of P_1P_3, P_2P_4, $\cdots$, P_5P_2. Prove that the lines A_5M_1, A_1M_2, $\cdots$, A_4M_5 are concurrent at a point O, such that the vectors OA_1, OA_2, $\cdots$, OA_5 form a closed polygon.

3386. *Proposed by H. E. Trefethen, Colby College.*

If the incircle passes through the centroid of a triangle, find positive integral values for the sides a, b, c.

3387. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

With the same vertex of a given triangle as center, two circles are drawn respectively orthogonal to the nine-point circle and to the conjugate circle of the triangle. Show that the ratio of the squares of the radii of the two circles is 1:2.

3388. *Proposed by S. A. Corey, Des Moines, Iowa.*

Prove that, if the tenth and higher derivatives of f vanish identically, $f(i)+f(-i) = 120f(0)+30f(r)+30f(-r)+640f(s)+640f(-s)-405f(s)-405f(-s)-324f(t) -324(-t)$, where $i=\sqrt{(-1)}$, $r=\sqrt{\frac{1}{2}}$, $s=\sqrt{\frac{1}{3}}$, $t=\sqrt{\frac{2}{3}}$, and where f is analytic. It follows that when the tenth and higher derivatives are negligibly small the indicated relation holds approximately.

SOLUTIONS

3020 [1923, 206]. *Proposed by John Nichols, Portland, Oregon.*

Rationalize $\alpha^{1/2}+\beta^{1/2}+\cdots+\mu^{1/2}+\nu^{1/2}+p=0$.

I. *Solution by C. K. Robbins, Purdue University.*

Consider $a^{1/2}+b^{1/2}+c^{1/2}+p=0$. This can be rationalized by the usual process.

Consider $a^{1/2}+b^{1/2}+c^{1/2}+d^{1/2}+p=0$. This becomes $-u+v=0$ by letting $a^{1/2}+b^{1/2}+c^{1/2}+u=0$ and $d^{1/2}+p=v$. Rationalize these two equations, the rationalized form of the first one being obtained from the previous case by replacing p by u. Then eliminate u and v from the three equations, using Sylvester's method where convenient.

It is obvious how to continue. The algebraic difficulties seem to be inherent in the problem.

II. *Solution by Otto Dunkel, Washington University.*

Set

$$x = \sum p_i^{1/2}, \quad \text{where} \quad = 1,2,\cdots,n; \tag{1}$$

and let us assume that no one of the radicals $p_i^{1/2}$ can be expressed rationally in terms of the other radicals and the p's including p_i. This excludes the case of any p being zero, or any radical having a rational value. A rational function of any set of quantities shall mean here a polynomial in these quantities with integral coefficients, or the quotient of two such polynomials. Suppose that x satisfies an irreducible equation $f(x)=0$ of degree m where the left side of the equation is a polynomial in x with coefficients which are rational functions of the p's and the coefficient of x^m is unity. Write $x=P_i+p_i^{1/2}$, where P_i is the sum of the remaining radicals, and insert this value of x in $f(x)=0$. Separating the terms of the result containing even powers of $p_i^{1/2}$ and those containing odd powers, we may write this result $M_i+N_ip_i^{1/2}$, where M_i and N_i are rational functions of the remaining radicals and the p's. If now $N_i\neq 0$, then $p_i^{1/2}$ can be expressed rationally in terms of the remaining radicals and the p's contrary to hypothesis. Hence $N_i=0$ and then $M_i=0$. It now follows that $P_i-p_i^{1/2}$ is also a root. Hence the equation $f(x)=0$ must have at least $N=2^n$ roots and therefore $m\geqq N$. It will now be shown that $m=N$ by constructing the equation. Let $r_1, r_2, \cdots, r_N$ be the N different values of (1) when the radicals in (1) are assinged the N different arrangements of $+$ and $-$ signs. It will be shown that

$$f(x) = \prod_{i=1}^{N}(x-r_i) = \sum_{i=1}^{N} b_i x^{N-i}, \tag{2}$$

The coefficients of the polynomial in x to the right in (2) are the elementary symmetric functions of the r's. The interchange of two of the p's in the set of r's merely changes the order of the r's in the set. Hence these coefficients are symmetric functions of the n radicals. Moreover, if we change the sign of any one radical, say $p_i^{1/2}$, wherever it occurs in the set of r's, the set is unchanged except as to order, and the coefficients of (2) are unchanged. Any one coefficient may be written $A_i+B_ip_i^{1/2}$, where A_i contains all the terms of this coefficient with even powers of $p_i^{1/2}$. Now if the sign of the radical $p_i^{1/2}$ alone is changed, the coefficient is unaltered in value and also A_i and B_i remain unchanged. Hence $A_i+B_ip_i^{1/2}=A_i-B_ip_i^{1/2}$, and therefore $B_i=0$, and the coefficient contains

no odd powers of $p_i^{1/2}$. Hence each coefficient is a rational integral function of the p's. Moreover it is symmetric in the p's and it can therefore be expressed as a polynomial in the elementary symmetric functions $a_1, a_2, \cdots, a_n$ where $a_i = \sum p_1 p_2 \cdots p_i$.

Some methods of computation: The computation of the coefficients of (2) by the direct multiplication of the linear factors is tedious even for small values of n, say $n=4$. For $n=2$ this direct method is rather easy. The following equations for $n=3, 4, \cdots$, may then be formed in turn as follows. Having found the equation $f(x)=0$ for n, write $f(x+q)=f_1(x, q^2)+qf_2(x, q^2)$, in which the even and odd powers of q have been separated as indicated. Then the equation for $n+1$ is

$$[f_1(x, p_{n+1})]^2 = p_{n+1}[f_2(x, p_{n+1})]^2.$$

Here additional labor would be required if we wished to reduce the coefficients to the elementary symmetric functions of $p_1, p_2, \cdots, p_{n+1}$.

This additional labor may be avoided in the following method, which appears to possess other advantages. Let

$$\phi(x) = x^n - \alpha_1 x^{n-1} + \alpha_2 x^{n-2} - \cdots = 0$$

be the equation whose roots are the n radicals $p_i^{1/2}$. Write $\phi(x)=\phi_1(x^2)+x\phi_2(x^2)$, where the even and odd powers of x have been separated. Then $[\phi_1(x)]^2 = x[\phi_2(x)]^2$ is the equation whose roots are $p_1, p_2, \cdots, p_n$, and whose coefficients are $a_1, a_2, \cdots, a_n$ with suitable signs. After expansion we find

$$a_{2i} = \overset{2}{\alpha}_{2i} + 2\sum \alpha_{2k}\alpha_{4i-2k} - 2\sum \alpha_{2k-1}\alpha_{4i-2k+1},$$
$$a_{2i-1} = \overset{2}{\alpha}_{2i-1} - 2\sum \alpha_{2k}\alpha_{4i-2k-2} + 2\sum \alpha_{2k-1}\alpha_{4i-2k-1}.$$

The case for $n=4$ will suffice to indicate the process. After suitable multiplication of each side and setting $\alpha_1 = x$, the equations for this case may be written

$$2\alpha_2 = x^2 - a_1, \qquad 4a_2 = 4\alpha_2^2 + 8\alpha_4 - 8\alpha_3 x, \qquad 64a_3 x^2 = 64(\alpha_3 x)^2 - 128\alpha_2\alpha_4 x^2.$$

By elimination of $\alpha_3 x$ from the last two equations we obtain a single equation involving α_2 and α_4, and we may set the terms containing even powers of α_4 on the left and the odd powers on the right. Square both sides and replace α_4^2 by a_4, and there results

$$\{[(x^2 - a_1)^2 - 4a_2]^2 - 64(a_3 x^2 - a_4)\}^2 - 256a_4\{3x^4 - 2a_1 x^2 + 4a_2 - a_1^2\}^2 = 0.$$

It is obvious that by setting $a_4=0$ and omitting the last exponent 2 we obtain the equation for $n=3$, and so on.

A method will now be given for calculating the coefficients of the equation (2). Each b with an odd subscript is zero, and we may write $b_{2i} = (-1)^i \sum r_1^2 r_2^2 \cdots r_i^2$, where the summation is extended to half of the roots, the other half having the same values but opposite signs. Hence b_{2i} may be expressed as a polynomial in $s_1, s_2, s_3, \cdots$, where $s_i = \sum r_j^{2i}$, $j=1, 2, \cdots, \sigma = 2^{n-1}$.

Suppose that r_1 is the sum of the radicals each taken with the positive sign. Then r_1^{2i} may be developed by the multinomial theorem in terms of products of the radicals $p_1^{1/2}, p_2^{1/2}, \cdots$, and for the purpose of summation to find s_i we may omit all terms containing an odd power of $p_j^{1/2}$, since such terms cancel out in the summation. Thus s_i may be expressed in terms of symmetric functions of the p's, and these functions may then be expressed in terms of the a's. Inserting these final values of the s's in the expressions for b_{2i} we obtain the desired results. The method will be clear from the results for the first four coefficients.

$$-b_2 = s_1, \quad 2b_4 = s_1^2 - s_2, \quad -6b_6 = s_1^3 - 3s_1s_2 + 2s_3,$$

$$24b_8 = s_1^4 + 3s_2^2 - 6s_1^2s_2 + 8s_1s_3 - 6s_4, \quad s_i = \sum r_j^{2i}, \quad j = 1, 2, \cdots, \sigma = 2^{n-1}.$$

Then by summation we find

$$s_1 = \sigma a_1, \quad s_2 = \sigma(a_1^2 + 4a_2), \quad s_3 = \sigma(a_1^3 + 12a_1a_2 + 48a_3),$$

$$s_4 = \sigma(a_1^4 + 16a_2^2 + 24a_1^2a_2 + 256a_1a_3 + 1088a_4).$$

After substituting these values in the expressions above, we have

$$b_2 = -\sigma a_1, \quad b_4 = \sigma C_2 a_1^2 - 2\sigma a_2,$$

$$b_6 = -{}_\sigma C_3 a_1^3 + 2\sigma(\sigma - 2)a_1a_2 - 16\sigma a_3,$$

$$b_8 = {}_\sigma C_4 a_1^4 - \sigma(\sigma - 2)(\sigma - 3)a_1^2a_2 + 2\sigma(\sigma - 2)a_2^2 + 16\sigma(\sigma - 4)a_1a_3 - 272\sigma a_4,$$

where ${}_\sigma C_i$ is the coefficient of x^i in the expansion of $(1+x)^\sigma$.

3332 [1928, 377]. *Proposed by R. E. Gaines, University of Richmond.*

In a given ellipse the two conjugate diameters are drawn which are equally inclined to the major axis, and a similar variable ellipse touches these two diameters and has its major axis on the same line as that of the fixed ellipse. Find the distance between their centers when the area common to the two ellipses is a maximum.

Solution by the Proposer.

The equations of the two ellipses may be written

$$b^2x^2 + a^2y^2 = a^2b^2, \quad (x - 2^{1/2}az)^2b^2 + a^2y^2 = a^2b^2z^2, \tag{1}$$

where the distance between the centers is $2^{1/2}az$, and z is the parameter. Let u and v be the two segments into which the distance between the centers is divided by the common chord, so that $u+v=2^{1/2}az$. If $2A$ is the area in common, we shall compute A. For this purpose, we shall consider the two ellipses as having their centers at the origin, and we may write

$$ay_1 = b(a^2 - x^2)^{1/2}, \quad ay_2 = b(a^2z^2 - x^2)^{1/2}, \tag{2}$$

where $y_1(u) = y_2(v)$ is one half the common chord in their original position. Then

$$A = \int_u^a y_1 dx + \int_v^{az} y_2 dx,$$

$$\frac{dA}{dz} = -y_1(u)\frac{du}{dz} - y_2(v)\frac{dv}{dz} + \int_v^{az} \frac{dy_2}{dz}dx, \tag{3}$$

$$= -2^{1/2}ay_2(v) + \int_v^{az} abz(a^2z^2 - x^2)^{-1/2}dx,$$

where reductions are made by use of the two relations above. Now set $x = az\cos\theta$, $v = az\cos\psi$, then $y_2(v) = bz\sin\psi$, and we have

$$dA/dz = abz(\psi - 2^{1/2}\sin\psi).$$

For actual intersection we must have $2^{1/2} - 1 \leqq z \leqq 2^{1/2} + 1$. Since $2^{3/2}zu = a(1+z^2)$, we find that $2^{3/2}\cos\psi = 3 - z^{-2}$. As z increases from its lower limit to its upper limit, ψ passes from π to 0, and therefore dA/dz is at first positive, then vanishes for ψ_0, and finally remains negative until it is zero again when $\psi = 0$. Hence there is a single maximum for A given by $\psi_0 = 2^{1/2}\sin\psi_0$. It is found that $\psi_0 = 79°43'49''$ approximately, $z = .63299$, and $2^{1/2}az = .89519a$.

3335 [1928, 377]. *Proposed by Paul Wernicke, Washington, D. C.*

Assume, on the sides of a triangle ABC, the points L on BC, M on CA, N on AB. Find the condition for the existence of a real straight line dividing the three joins AL, BM, CN in the same ratio, and, in particular, bisecting them.

Solution by the Proposer.

In homogeneous coordinates ($A \equiv 1, 0, 0$; $B \equiv 0, 1, 0$; $C \equiv 0, 0, 1$), the coordinates of L, M, N, dividing BC, CA, AB in the respective ratios l, m, n are $(0, 1, l)$, $(m, 0, 1)$, $(1, n, 0)$. Then AL, BM, CN are divided in the ratio k in the points $(1+l, k, kl)$, $(km, 1+m, k)$, $(k, kn, 1+n)$. If these last three points are collinear, then

$$\begin{vmatrix} 1+l & k & kl \\ km & 1+m & k \\ k & kn & 1+n \end{vmatrix} = 0.$$

This equation reduces to $k+1 = 0$, giving the line at infinity, and to

$$(1 + lmn)k^2 - (1 + l)(1 + m)(1 + n)(k - 1) = 0.$$

There will thus be two real distinct lines of the required kind if the discriminant

$$(1 + l)(1 + m)(1 + n)[(1 + l)(1 + m)(1 + n) - 4(1 + lmn)].$$

is positive, and only one if it vanishes. This refers to the finite lines. The line at infinity is always of the required type.

If $k = 1$, the case of bisection, $lmn = -1$, and it then follows from the theorem

of Menelaus that L, M, N are collinear. Conversely, if L, M, N are collinear, $k=1$ is a solution. The others are $k=\infty$ for the straight line LMN, and $k=-1$ for the line at infinity.

Note by the Editors. The derivation of the coordinates of the points above is simplified by recalling that in this system of coordinates, if x, y, z are the coordinates of P, then P is the center of mass of a system of three particles at A, B, C of masses x, y and z.

3336 [1928, 378]. *Proposed by Otto J. Ramler, The Catholic University of America.*

Construct a triangle having given $a-b$, h_b+h_c, and the angle A, where a and b are sides, and h_b, h_c are the altitudes upon the sides b, c, respectively. (Altshiller-Court's College Geometry, p. 28, no. 7).

Solution by the Proposer.

The data are equivalent to having given $a-b$, $b+c$, $a+c$, and angle A, since angle A, h_b+h_c, and $b+c$ form a datum (Cf. "College Geometry" p. 28). We observe $a+c\equiv(b+c)+(a-b)$.

Construct a triangle $A''CE$, with $CE=a-b$, $A''C=a+c$, $A''CE=180°-A$. Produce EC to B' making $EB'=a+c$. Then $CB'=b+c$. Complete the parallelogram $A''EB'D$ so that D is opposite to E. Then $A''D=a+c$, and DB' is parallel to $A''E$. With C as center, and $a+c$ as radius strike an arc meeting DB' produced in A'. Let A be the intersection of $A''A'$, EB'. Through A draw a line parallel to $A''C$ meeting CA' at B. Then triangle CAB is the required triangle.

For the angle $CAB=A''CA=A$, because AB is parallel to $A''C$. Since triangle $A''CA'$ is isosceles, so also is triangle ABA'. Whence, $AB=BA'$. The angle $DA''C=B'AB$ (sides are parallel). And $AB:A''C=AA':A''A'=AB':A''D$. But $A''D=A''C$. Whence, $AB'=AB$. Therefore, $AB'=AB=BA'$, and $CB'=CA+AB=b+c$, $CA'=CB+AB=a+c$, and angle $CAB=A$.

Therefore, triangle ABC satisfies all conditions imposed. The problem has one and only one solution if we consider the parts of the triangle to be positive only.

Also solved by Paul Wernicke.

3337 [1928, 378]. *Proposed by E. B. Escott, Oak Park, Illinois.*

Sum the series

$$\frac{6^3}{2\cdot3\cdot4\cdot5}x^5-\frac{8^3}{4\cdot5\cdot6\cdot7}x^7+\frac{10^3}{6\cdot7\cdot8\cdot9}x^9-\cdots.$$

I. *Solution by Robert E. Moritz, University of Washington.*

The series is obviously absolutely convergent for all values of x for which $|x|<1$, conditionally convergent for $x=1$, and divergent when $|x|>1$. Let us first restrict ourselves to values of x such that $|x|<1$.

The general term of the given series is

$$u_{n-2} = (-1)^{n-1}(2n)^3x^{2n-1}/(2n-4)(2n-3)(2n-2)(2n-1)$$

$$= u^{(1)}_{n-2} + u^{(2)}_{n-2} + u^{(3)}_{n-2} + u^{(4)}_{n-2},$$

where

$$u^{(1)}_{n-2} = (-1)^{n-1}(-1/6)x^{2n-1}/(2n-1), \quad u^{(2)}_{n-2} = (-1)^{n-1}(4)x^{2n-1}/(2n-2),$$

$$u^{(3)}_{n-2} = (-1)^{n-1}(-27/2)x^{2n-1}/(2n-3), \quad u^{(4)}_{n-2} = (-1)^{n-1}(32/3)x^{2n-1}/(2n-4).$$

Now the original series being absolutely convergent for the values of x under consideration, its sum may be found by adding the sums of the series whose terms are $u^{(1)}_{n-2}$, $u^{(2)}_{n-2}$, $u^{(3)}_{n-2}$, and $u^{(4)}_{n-2}$ respectively, the initial value of n in each case being 3. Now it is easily seen that when $|x|<1$,

$$\sum_3^\infty u^{(1)}_{n-2} = -\tfrac{1}{6}[\tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \tfrac{1}{9}x^9 - \cdots] = -\tfrac{1}{6}[\tan^{-1}x - x + \tfrac{1}{3}x^3],$$

$$\sum_3^\infty u^{(2)}_{n-2} = 4x[\tfrac{1}{4}x^4 - \tfrac{1}{6}x^6 + \tfrac{1}{8}x^8 - \cdots] = +2x[x^2 - \log(1+x^2)],$$

$$\sum_3^\infty u^{(3)}_{n-2} = (-27x^2/2)[\tfrac{1}{3}x^3 - \tfrac{1}{5}x^5 + \tfrac{1}{7}x^7 - \cdots] = (-27x^2/2)[x - \tan^{-1}x];$$

$$\sum_3^\infty u^{(4)}_{n-2} = (32x^3/3)[\tfrac{1}{2}x^2 - \tfrac{1}{4}x^4 + \tfrac{1}{6}x^6 - \cdots] = (16x^3/3)\log(1+x^2).$$

Adding, we find the sum of the given series, viz.,

$$(A) \quad \sum_3^\infty u_{n-2} = \tfrac{1}{6}x - (104x^3/9) + \tfrac{1}{6}(81x^2-1)\tan^{-1}x + \tfrac{2}{3}x(8x^2-3)\log(1+x^2).$$

Finally, by a theorem of Abel (Crelle's Journal, vol. 1, 1826), since the series is known to be convergent when $x=1$, its sum as x approaches 1 will also be given by (A); in short, whenever the given series has a sum its value will be given by (A).

II. *Solution by Mildred Barr, Iowa State College.*

Let $f(x)$ denote the given series. We shall proceed formally and differentiate it four times with respect to x, and we find that

$$f^{iv}(x) = 8\sum(-1)^{n+1}(n+2)^3x^{2n-1}.$$

We multiply both sides of the above equation by $(1+x^2)^4$, and find that the coefficient of x^{2n-1} for $n>4$ is

$$(-1)^{n+1}8[(n+2)^3 - 4(n+1)^3 + 6n^3 - 4(n-1)^3 + (n-2)^3] = 0.$$

Therefore

$$f^{iv}(x) = 8(27x + 44x^3 + 31x^5 + 8x^7)/(1+x^2)^4.$$

We integrate this expression for $f^{iv}(x)$ four times, determine the constants of integration by letting $x=0$ in the successive integrals, and we obtain

$$f(x) = \frac{2x(8x^2-3)}{3}\log(1+x^2) + \frac{81x^2-1}{6}\tan^{-1}x - \frac{104x^3}{9} + \frac{x}{6}.$$

It is obvious that the foregoing processes are valid when $-1<x<1$.

Also solved by E. H. Clarke, M. S. Knebelman, J. F. Reilly, and the Proposer.

3338 [1928, 378]. *Proposed by N. A. Court, University of Oklahoma.*

The two pairs of extremities of two harmonic segments determine an involution in which the mid-points of the segments are two conjugate points.

Solution by Paul Wernicke, Washington, D. C.

In Cartesian coordinates, let the abscissae of the midpoints be x, u. Then the abscissae of the pairs may be written $x \pm y$ and $u \pm v$. Let the double points of their involution have the abscissae a and b, then we have

$$(x+y-a)(x-y-b) + (x+y-b\,(x-y-a) = 0,$$

or

$$x^2 - y^2 - (a+b)x + ab = 0.$$

Also,

$$u^2 - v^2 - (a+b)u + ab = 0.$$

The condition that the pair of midpoints belong to the same involution is

$$(x-a)(u-b) + (x-b)(u-a) = 0,$$

or

$$2xu - (a+b)(x+u) + 2ab = 0,$$

which will be a consequence (namely the sum) of the preceding two equations if we can show that $2xu = x^2 - y^2 + u^2 - v^2$.

This last equation is merely a way of writing the condition,

$$(x+y-u-v)(x-y-u+v) + (x+y-u+v)(x-y-u-v) = 0,$$

that the given pairs be harmonic.

Note by the Editors. This theorem is obvious from geometrical considerations. If A, A'; B, B' are four points on a straight line such that the first pair of points is harmonically separated by the second pair, the two circles with diameters AA' and BB' cut orthogonally in a point O. If C and C' are the centers of the two circles, AOA', BOB', COC' are right angles and A, A'; B, B'; C, C' are pairs of points in the involution thus determined by O. The converse is easily proved in the same manner.

3339 [1928, 378]. *Proposed by Paul Capron, U. S. Naval Academy.*

Discuss the nodes of the parabolic spiral $(\rho-a)^2 = ca\theta$.

Solution by the Proposer.

The two branches of the parabolic spiral are given by the pair of equations $\rho = a[1 \pm (c\theta/a)^{1/2}]$, and there is a node $(\rho_{1,2}, \theta_{1,2})$, given by their intersection, when and only when

$$\theta_2 = (2k+1)\pi + \theta_1 \quad (k \text{ integral}), \tag{1}$$

and

$$-\rho_2 = \rho_1. \tag{2}$$

Hence $(c\theta_2/a)^{1/2} - (c\theta_1/a)^{1/2} = 2$, or $m - n = 2$, if we set $m^2 = c\theta_2/a$, $n^2 = c\theta_1/a$. From (1) we have

$$m^2 - n^2 = (2k+1)\pi c/a = (2k+1)l, \quad \text{where} \quad l = \pi c/a. \tag{3}$$

Solving the above pair of equations in m and n, we obtain as the conditions for a node $4(n+1) = 4(m-1) = (2k+1)l$. Hence nodes occur at the points given by

$$\theta_1 = n^2\pi/l, \qquad \rho_1 = (n+1)a, \qquad \theta_2 = m^2\pi/l, \qquad \rho_2 = (1-m)a$$

provided, k being any integer, that

$$n = (2k+1)l/4 - 1, \qquad m = (2k+1)l/4 + 1, \qquad l = \pi c/a.$$

The increment of θ between two successive nodes (ρ, θ) and (ρ', θ'), given by $n = \frac{1}{4}(2k+1)l - 1$, $n' = \frac{1}{4}(2k+3)l - 1$, is

$$\theta' - \theta = (n'^2 - n^2)\pi/l = (n' - n)(n' + n)\pi/l = [(k+1)l - 2]\pi/2.$$

3341 [1928, 445]. *Proposed by Paul Wernicke, Washington, D. C.*

Given a triangle ABC and a point P, find the conditions for the construction of a triangle having its sides parallel and proportional to the joins PA, PB and PC.

Solution by J. Rosenbaum, Milford, Conn.

It is easily seen that there is no loss in generality if the proportionality factor is taken as unity. We accordingly take a point P in the plane ABC, and assuming that for this position of P the construction is possible, we draw PA and then draw AD parallel and equal to PB, and then draw DP.

Since the triangle PAD is supposed to fulfill the requirements, CPD is a straight line with point P lying between C and D. Also since AD is parallel and equal to PB, the figure $ADBP$ is a parallelogram, and hence PD bisects AB.

It thus follows that for a construction to be possible, it is necessary that P lie on the median to AB, and since AB is any side of ABC, the point P has to lie on all the three medians. In other words P has to be the centroid of the triangle ABC.

It is easily proved that this condition is also sufficient.

It is interesting to note that the analogous point in a quadrilateral is the intersection of the lines which join the midpoints of the two pairs of opposite sides (which point is not necessarily the center of area).

Also solved by Rufus Crane, Laurence Hampton, R. A. Johnson, Harry Langman, Enrique Linares, A. Pelletier, and the Proposer.

Note by the Editors. This solution considers a direction requirement which may be stated thus: The position of P is to be determined so that the forces defined in direction and magnitude by PA, PB, and PC shall be in equilibrium. If the direction requirement is dropped, or altered, there are three other solutions given by the vertices of the triangle formed by the three straight lines through the vertices of the triangle parallel to the opposite sides.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

Professor G. D. Birkhoff, of Harvard University, has been elected foreign associate of the Royal Academy dei Lincei.

Professors Marston Morse and J. L. Walsh, of Harvard University, have been elected fellows of the American Academy of Arts and Sciences.

Honorary doctorates have been conferred on Professor A. H. Compton, of the University of Chicago, by Yale University and by The Ohio State University.

The University of Michigan has conferred the honorary degree of doctor of laws on Professor R. A. Millikan, of the California Institute of Technology.

Harvard University has conferred an honorary doctorate on Professor H. N. Russell, of Princeton University.

Swarthmore College has conferred an honorary doctorate on Dr. W. F. G. Swann, director of the Bartol Foundation of the Franklin Institute.

Marietta College has conferred the honorary degree of doctor of science on Dr. Frank C. Jordan, professor of astronomy in the University of Pittsburgh.

The board of trustees of Monmouth College has determined that the most fitting tribute which can be given to Miss Alice Winbigler for the great service she has rendered to Monmouth College during a period of fifty years' continuous teaching is the establishment of a permanent endowment for an Alice Winbigler chair of mathematics in the college. It is proposed to raise the sum of $60,000. A retiring allowance of $1500 each year, which will be approximately one half the income from this endowment, is to go to Miss Winbigler during her lifetime, after which the entire income will be used for the permanent support of the chair of mathematics.

Dr. Charles S. Howe retired from the presidency of the Case School of Applied Science in August.

The corner stone of the new Eckhart Hall on the main quadrangle of the University of Chicago was laid on July 12. This Hall will be occupied by the departments of physics, mathematics, and astronomy.

Associate Professor B. A. Bernstein, of the University of California, has been promoted to a professorship of mathematics.

Professor Florian Cajori, of the University of California, has retired, as professor emeritus of the history of mathematics.

Associate Professor C. C. Camp, of the University of Nebraska, has been promoted to a professorship.

Assistant Professor E. F. A. Carey, of the University of Montana, has been promoted to an associate professorship.

Associate Professor J. C. Fitterer, of the Colorado School of Mines, has been promoted to a professorship.

Dr. William Gafafer has been appointed research associate at Johns Hopkins University.

Professor W. V. N. Garretson, of Ouchita College, Arkadelphia, has been appointed associate professor of mathematics at the Oklahoma Agricultural and Mechanical College.

Assistant Professor J. W. Hurst, of Montana State College, has been promoted to an associate professorship.

Dr. C. G. Jaeger, of the University of Missouri, has been appointed to an assistant professorship at Tulane University.

Miss Elizabeth Knight, of the Milwaukee State Normal School, has been promoted to an assistant professorship.

Mr. T. R. Long has been promoted to an assistant professorship at the University of Rochester.

Associate Professor C. E. Love has been promoted to a professorship at the University of Michigan.

Mr. C. I. Lubin, of the University of Cincinnati, has been promoted to an assistant professorship.

Mr. C. T. Male, of Union College, has been promoted to an assistant professorship.

Dr. Earl S. Mickelson, of the University of Minnesota, has been appointed head of the department of Mathematics and physics at the New Mexico State Teachers' College, Silver City, N. M.

Dr. G. W. Myers, professor of the teaching of mathematics and astronomy in the School of Education at the University of Chicago, has retired.

Mr. G. A. Parkinson, of the University of Wisconsin, has been promoted to an assistant professorship.

Mr. C. S. Porter, of Amherst College, has been promoted to an associate professorship.

Dr. Tibor Rado has been appointed lecturer in mathematics at Harvard University for the first term of 1929–30. He will give courses on minimal surfaces and the problem of Plateau, and on the elementary theory of differential equations.

Assistant Professor H. P. Robertson, of the California Institute of Technology, has been appointed assistant professor of mathematical physics at Princeton University.

Dr. William S. Slauch, chairman of the department of mathematics of the High School of Commerce of New York City, has been appointed assistant professor of mathematics at New York University.

Dr. I. M. Sheffer, National Research Fellow, has been appointed assistant professor of mathematics at Pennsylvania State College.

Assistant Professor J. H. Taylor, of the University of Wisconsin, has been appointed professor of mathematics at George Washington University.

Mr. V. C. D'Unger, formerly of Little Rock College, is now auditor and general office manager of the Pyramid Life Insurance Company of Little Rock.

Assistant Professor R. L. Wilder, of the University of Michigan, has been promoted to an associate professorship.

Mr. F. G. Williams, of Cornell University, has been appointed professor of mathematics at Susquehanna University.

The following appointments to instructorships are announced:

Columbia University, Dr. A. A. Albert.

College of the City of Detroit, Dr. K. W. Folley, Dr. D. C. Morrow.

Harvard University, Dr. T. F. Cope, full time; Mr. A. E. Anderson, Mr. A. E. Currier, Mr. G. A. Hedlund, Mr. S. H. Kimball, Mr. C. B. Morrey, Mr. C. S. B. Myers, Mr. G. Saute, Mr. G. B. Van Schaack, Mr. C. Wexler, part time.

University of Iowa, Dr. J. M. Earl.

Long Island University, Mr. E. A. Knobelauch.

Purdue University, Mr. E. L. Klinger, Mr. J. M. Thompson.

Rice Institute, Dr. L. M. Blumenthal.

Worcester Polytechnic Institute, Mr. S. A. Lepeshkin.

The Chauvenet Prize

In the year 1925, the ASSOCIATION established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association.

The purpose of the prize is to stimulate expository contributions in mathematical journals. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included. They carry their own reward in the form of a liberal cash honorarium to each author.

It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the CHAUVENET PRIZE will tend to stimulate such production.

The retiring President of the Association, Professor W. B. Ford, has given an additional endowment for this prize whereby it will hereafter be awarded every three years. The next award, however, will be in December, 1929, for the period 1925-1928 inclusive.

Note that the prize is to be awarded only to a *member* of the ASSOCIATION—one more of the many good reasons for membership.

Legal Form for Gifts and Bequests

I hereby give[1] to the Board of Trustees of the Mathematical Association of America the sum of ..... Dollars, to be known as the ..... Fund, and to be used for[2] { Endowment—the income only of which may be expended. / Special Projects—for which both principal and income may be expended. }

Witness: ..... Signature .....

[1] In case of a bequest, the first line should read "I hereby give and bequeath," etc.
[2] Indicate which one of the two purposes is desired, and omit the other.

The Association needs funds for scientific publications and for the promotion of scientific activities.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to R. A. JOHNSON, Hunter College, New York, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Thirteenth Summer Meeting of the Association, Boulder, Colorado, August 26-27, 1929.
Fourteenth Annual Meeting, Des Moines, Iowa, December 31, 1929, January 1, 1930.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1929.

ILLINOIS, Carthage, Ill., May 3-4.
INDIANA, Culver Military Academy, May 3-4.
IOWA, Fairfield, Iowa, April 26-27.
KANSAS, Topeka, Kansas, February 2.
KENTUCKY, Lexington, Ky., April 13.
LOUISIANA-MISSISSIPPI, Lafayette, La., April 12-13.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, George Washington University, May 4.
MICHIGAN, Ann Arbor, Mich., March 16.
MINNESOTA, St. Paul, Minn., May 11.
MISSOURI, Kansas City, Mo., November 16.
NEBRASKA.
OHIO, Columbus, Ohio, April 4.
PHILADELPHIA, University of Pennsylvania, November 30.
ROCKY MOUNTAIN, Greeley, Colo., April 12-13.
SOUTHEASTERN, Macon, Ga., April 19-20.
SOUTHERN CALIFORNIA, University of Redlands, March 9.
TEXAS, Houston, Texas, Jan. 26.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

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AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

THE RHIND MATHEMATICAL PAPYRUS

At last we are able to announce with confidence that the second volume of the Rhind Papyrus is actually in process of completion and that the two volumes are expected to be ready for delivery not later than December first, 1929. The long delay, while unavoidable, has nevertheless contributed to the scientific perfection of the work and has permitted the addition of certain valuable features, especially information concerning the Golenishchev mathematical papyrus in Moscow, and an official account and photograph of the Mathematical Leather Roll, British Museum 10250, a document of the same period as the Rhind Papyrus.

A full description of the Chace publication of the Papyrus was printed in this MONTHLY for August–September, 1926, and ample reasons will there be found to justify the purchase of these volumes for every college library and for many of the larger high schools. The price there stated of fifteen dollars, postpaid, to members of the Association is considerably less than half of the actual cost of preparation and publication, but it was the desire of Doctor Chace that the Association members should be thus favored and he further showed his loyalty to the Association by making an outright gift to it of practically the whole edition with the understanding that the income from the sales should be kept as an endowment fund of the Association.

The Trustees, therefore, feel that they should deposit in that fund the full fifteen dollars for every set sold, and while they will surely fulfill the printed promise to prepay the carrying charges to the members who have already subscribed, nevertheless these will confer a favor and do a service to the Association by returning those charges upon receipt of the volumes. The parcel post charge will be well toward a dollar on an average, according to the distance carried. However, to those members who have not yet subscribed, notice is hereby given that on and after December first, 1929, the price will be fifteen dollars *plus carrying charges* to members who subscribe thereafter, directly through the Secretary of the Association.

This membership price will be extended to all new institutional and individual members whose applications are received by the Secretary on or before December first.

It is understood that the distribution to all non-members will be through the Open Court Publishing Company, 339 East Chicago Avenue, Chicago, Illinois, and that the price will be twenty dollars plus carriage charges.

Those of us who have seen the proof sheets of the plates in the second volume, containing the complete photographic reproduction of the Papyrus, the presentation of the hieratic text in its original colors, the hieroglyphic transcription, the transliteration and literal translation, can only marvel at the extent and character of the work which Doctor Chace and his assistants have accomplished. Likewise, the contents and make-up of the first volume completed two years ago command our admiration. This volume contains 48 pages